G-charge Thouless pumps

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Quantum Hall effect and Topological phases

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Thouless pump

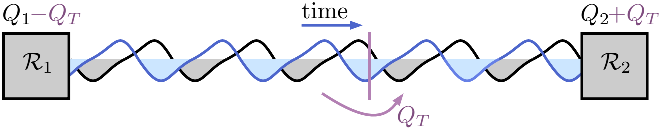
- ▷ Non-interacting particles in a periodic-in-time potential

$$H(t) = -\frac{d^2}{dx^2} + V(x,t) \qquad V(x, t+T) = V(x,t)$$

- ▷ Thouless (1983): If
  1. the driving is slow (adiabatic)
  2. the Fermi level is in a gap for all $t \in [0,T]$

  then

$$\langle Q_T \rangle \in \mathbb{Z}$$



- ▷ From 0 to T: loop of Fermi projections

Interacting setting

- Quantum spin chain, with algebra

$$\mathcal{A} = \otimes_{x \in \mathbb{Z}} \mathcal{A}_x \qquad \mathcal{A}_x \simeq M_{n \times n}(\mathbb{C})$$

- State as a normalized linear functional

$$\psi : \mathcal{A} \to \mathbb{C} \qquad \psi(\mathbb{1}) = 1$$

- Symmetry: Compact group G and a continuous action on $\mathcal{A}_x$:

$$\gamma : G \times \mathcal{A}_x \to \mathcal{A}_x, \qquad A \mapsto \gamma_g(A)$$

  such that

$$\gamma_g \circ \gamma_h = \gamma_{gh}$$

  $\rightsquigarrow$ extends to all $\mathcal{A}$

G-invertible state

- ψ is G-invariant

$$\psi(\gamma_g(A)) = \psi(A)$$

- ψ is invertible (Kitaev): There is a product state ω, a state $\bar{\psi}$ s.t.

$$\psi \otimes_{\text{aux}} \bar{\psi} \sim \omega$$

No long range entanglement
Equivalence: $\phi \sim \phi'$ if

$$\phi'(A) = \phi(\tau_1^H(A))$$

where τ_1^H is a Hamiltonian evolution by $H(s)$ over $s \in [0, 1]$

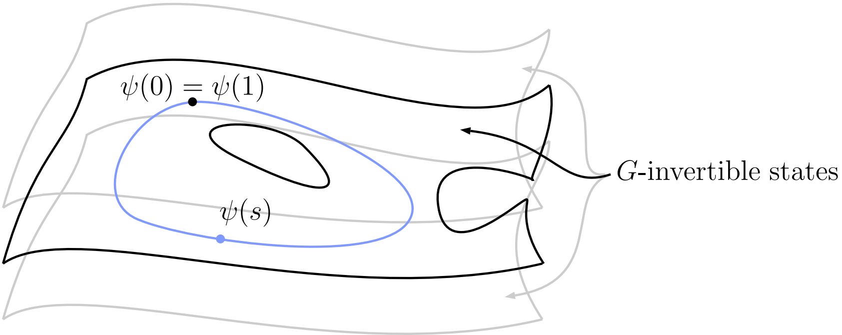
- ψ is G-invertible if it is G-invariant and invertible with

$$\gamma_g(H(s)) = H(s)$$

Classification

- Classification of G-invertible states in 1d:
  (Chen-Gu-Wen 2011, Ogata 2021, Kapustin-Sopenko-Yang 2021)
  Equivalence classes labelled by projective representations of G
- Adiabatic pumps: classify loops of G-invertible states:

$$\psi(s), s \in [0,1] : \psi(0) = \psi(1)$$

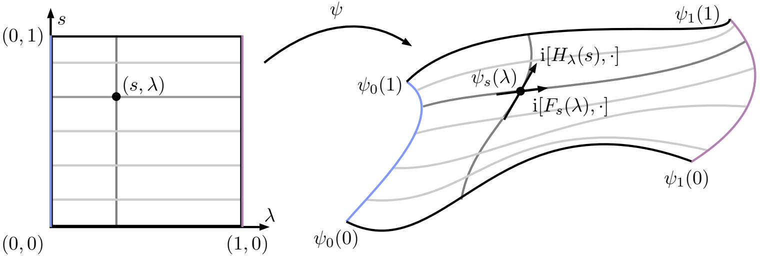


Classification of pumping

- What is a loop of states? No good topology so

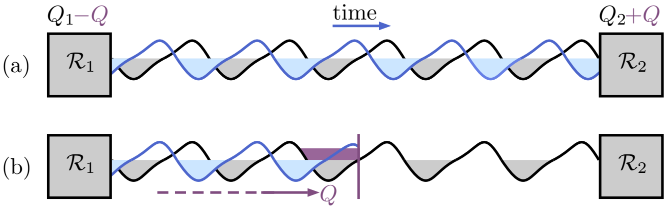
$$\psi(s)(A) = \psi(0)(\tau_s^K(A))$$

  for time-dependent Hamiltonian $K(s), s \in [0,1] \rightsquigarrow$ local continuity
  A G-loop requires $\gamma_g(K(s)) = K(s)$
- Equivalence of paths (hence loops)? Local homotopy



The index: un jeu d'enfant

Pumping charge



Consider the truncated Hamiltonian $K_-(s)$ acting only on $(-\infty, 0]$ and the pumped state

$$\psi_-(s) = \psi(0) \circ \tau_s^{K_-}$$

In general:

$$\psi(1) = \psi(0) \quad \text{but} \quad \psi_-(1) \neq \psi(0)$$

Expect: $\psi_-(1)$ carries a G-charge pumped from $-\infty$

Cannot be removed locally

The pumped state

Because of

1. short range entanglement,
2. locality of the dynamics along the loop,

and $\psi(1) = \psi(0)$, we have

$$\lim_{r\to\infty} \|\psi_-(1)(A_r) - \psi(0)(A_r)\| = 0 \qquad \mathrm{supp}(A_r) \cap [-r, r] = \emptyset$$

$\rightsquigarrow$ $\psi_-(1)$ and $\psi(0)$ are equal at infinity;

equivalently, $\psi_-(1)$ and $\psi(0)$ only differ along the cut

Mathematical consequence: $\psi_-(1)$ and $\psi(0)$ are unitarily equivalent

GNS representations and all that

GNS representation of any state ω on the chain algebra $\mathcal{A}$:

$$(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$$

such that

$$\omega(A) = \langle \Omega_\omega | \pi_\omega(A) | \Omega_\omega \rangle$$

In general, two states ω, ν cannot be represented in the same Hilbert space:

$$\nu(A) \stackrel{?}{=} \langle \Psi_\nu | \pi_\omega(A) | \Psi_\nu \rangle \qquad |\Psi_\nu\rangle = U|\Omega_\nu\rangle$$

In fact: $U : \mathcal{H}_\nu \to \mathcal{H}_\omega$ exists if and only if ω, ν are equal at infinity

Pumping index

Hence, $\psi_-(1)$ has a vector representative in GNS of initial state $\psi(0)$

$$|\Psi_{\psi_-(1)}\rangle \in \mathcal{H}_{\psi(0)}$$

$\rightsquigarrow$ 0-dimensional problem now

Unitary representation $U(g)$ of γ_g in GNS of G-invariant $\psi(0)$:

$$\pi_{\psi(0)}(\gamma_g(A)) = U(g)^* \pi_{\psi(0)}(A) U(g)$$

By G-invariance of $\psi_-(1), \psi(0)$:

$$U(g)|\Psi_{\psi_-(1)}\rangle = \mathrm{e}^{\mathrm{i}h(g)}|\Psi_{\psi_-(1)}\rangle \qquad U(g)|\Omega_{\psi(0)}\rangle = |\Omega_{\psi(0)}\rangle$$

Loop index:

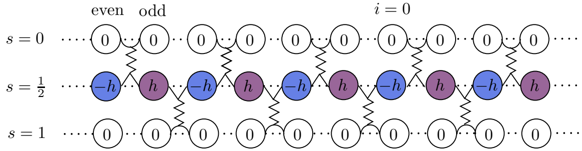
$$\mathrm{Ind}(\psi) = h(\cdot) \in \mathrm{Hom}(G, \mathbb{S}^1)$$

Classification of G-Thouless pumps

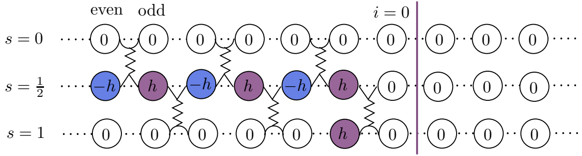
Theorem [B-De Roeck-Fraas-Jappens]
For a compact group G,

1. *If ψ is a constant loop, then $\mathrm{Ind}(\psi) = 0$*
2. *For any $h \in \mathrm{Hom}(G, \mathbb{S}^1)$, there is a loop such that $\mathrm{Ind}(\psi) = h$*
3. *If ψ_1, ψ_2 are homotopic, then $\mathrm{Ind}(\psi_1) = \mathrm{Ind}(\psi_2)$*
4. *If ψ_1, ψ_2 have G-equivalent basepoints and $\mathrm{Ind}(\psi_1) = \mathrm{Ind}(\psi_2)$, then ψ_1, ψ_2 are homotopic*
5. *Additivities:*
   - *Stacking:* $\mathrm{Ind}(\psi_1 \otimes_{\mathrm{aux}} \psi_2) = \mathrm{Ind}(\psi_1) + \mathrm{Ind}(\psi_2)$
   - *Concatenation:* $\mathrm{Ind}(\psi_1 \boxplus \psi_2) = \mathrm{Ind}(\psi_1) + \mathrm{Ind}(\psi_2)$

Example



(a) Time evolution generated by H



(b) Time evolution generated by the left-restricted H'

Pick $h \in \mathrm{Hom}(G, \mathbb{S}^1)$.
One-site Hilbert space

$$\mathcal{H} = \mathrm{span}\{|-h\rangle, |0\rangle, |h\rangle\}$$

with group action

$$U(g)|\tilde{h}\rangle = \mathrm{e}^{\mathrm{i}\tilde{h}(g)}|\tilde{h}\rangle$$

$$(\tilde{h} = -h, 0, h; \quad g \in G)$$

Classification of G-Thouless pumps

Note:

- $\mathrm{Hom}(G, \mathbb{S}^1) = H^1(G, \mathbb{S}^1)$, the first cohomology group of G
- Projective representations are classified by $H^2(G, \mathbb{S}^1)$, the second cohomology group of G

In other words: The full classification of G-loops in 1d is given by

$$H^{1+1}(G, \mathbb{S}^1) \oplus H^{0+1}(G, \mathbb{S}^1)$$

namely

(basepoints, loops)

Contracting loops

Central claim:

If $\mathrm{Ind}(\psi) = 0$ and $\psi(0)$ is a product, then ψ is a contractible loop

Sketch:

1. Split into loops on finite intervals of the chain
2. Contract each factor
3. Contract remaining short loop

Splitting a loop once

$$\psi(s) = \psi(0) \circ \tau_s^K$$

- ▷ First split $K = K_- + K_+ + K_0$, with $\mathrm{supp}(K_\pm) \subset \mathbb{Z}_\pm$
- ▷ Since $\psi(0)$ is a product

$$\psi(0) \circ \tau_1^{K_- + K_+} = \psi_- \otimes \psi_+$$

where ψ_- is the pumped state (ψ_+ is its G-inverse)

There are a $\tilde{K}^\pm$ localized around $\{0\}$ such that

$$\psi_\pm = \psi(0)_\pm \circ \tau_1^{\tilde{K}^\pm}$$

Charge cannot be created locally:

Vanishing index $\Leftrightarrow$ $\tilde{K}^\pm$ can be chosen G-invariant

Splitting a loop

Conclusion:

$$\psi(0) \circ \tau_1^{K-K_0} = \psi_- \otimes \psi_+ = \psi(0) \circ \tau_1^{\tilde{K}_- + \tilde{K}_+}$$

New split, G-invariant loop

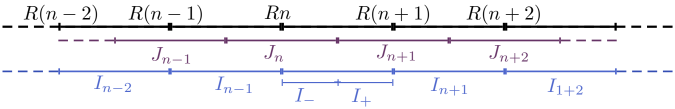
$$\psi(0) \circ \tau_1^{K-K_0} \circ (\tau_1^{\tilde{K}_- + \tilde{K}_+})^{-1} = \psi(0)$$

Summarizing: There is G-invariant $E_0(s)$ such that

1. Loop: $\psi(0) \circ \tau_1^{K+E_0} = \psi(0)$
2. Split loop: $K(s) + E_0(s)$ is split at $\{0\}$
3. Local perturbation: $E_0(s)$ supported around $\{0\}$

Splitting a loop

With points far apart,



the generator

$$K + \sum_n E_n$$

is almost split at all sites $Rn, n \in \mathbb{Z}$.

Correcting:

$$\alpha^K = \alpha^{\tilde{K}(R)} \circ \alpha^{W(R)} \circ \alpha^{\bar{K}(R)}$$

- ▷ $\tilde{K}(R), \bar{K}(R)$ split over finite intervals I_n, J_n generating loops
- ▷ $W(R)$ remainder of $\mathcal{O}(R^{-\infty})$

Contracting loops

Over any finite interval: Any loop of state is contractible, hence

$$\psi(0) \circ \alpha_s^K \quad \text{is homotopic to a short loop} \quad \psi(0) \circ \alpha_s^{W(R)}$$

Contracting the short loop:

- The product state $\psi(0)$ is a gapped ground state
- By perturbation theory, $\psi(0) \circ \alpha_s^{W(R)}$ are all gapped ground states
- Use parallel transport to contract the short loop

Higher dimensions?

- In d dimensions: The pumped state is an almost $(d-1)$-dimensional G-invertible state
- Natural conjecture: G-loops in d dimensions with a fixed basepoint have the same classification as $(d-1)$-dimensional G-invertible states
- For $d < 5$: $H^d(G, \mathbb{S}^1)$
- Technical challenge: $\rightsquigarrow$ no common GNS Hilbert space