Landau levels for Bochner Laplacian, Conference on quantum Hall effect and topological phases, Strasbourg, june 2022

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Landau level

The Landau Hamiltonian is the operator $\hat{H}$ acting on $L^2(\mathbb{R}^2)$

$$\begin{aligned} \hat{H} &= \tfrac{1}{2}(\pi_x^2 + \pi_y^2) \qquad \text{with } \pi_x = \tfrac{1}{i}\partial_x + \tfrac{B}{2}y,\ \pi_y = \tfrac{1}{i}\partial_y - \tfrac{B}{2}x \\ &= B(\mathfrak{a}^*\mathfrak{a} + \tfrac{1}{2}) \qquad \text{with } \mathfrak{a} = \tfrac{1}{\sqrt{2B}}(-\pi_y + i\pi_x),\ [\mathfrak{a}, \mathfrak{a}^*] = 1 \end{aligned}$$

The spectrum of $\hat{H}$ is $\{B(n + \frac{1}{2})/\ n \in \mathbb{N}\}$. Its eigenspaces are the Landau levels

$$\operatorname{Ker}(\hat{H} - B(n + \tfrac{1}{2})) = (\mathfrak{a}^*)^n \operatorname{Ker}(H - \tfrac{B}{2})$$

If we restrict $\hat{H}$ to $\operatorname{span}\{(\mathfrak{a}^*)^n|1\rangle,\ n \in \mathbb{N}\}$ with $|1\rangle = e^{-\frac{B}{4}(x^2+y^2)}$, same spectrum but simple eigenvalues.

Goal

- define Landau levels for Bochner Laplacian of compact manifold, understand influence of topology and geometry
- dimension of Landau levels in terms of characteristic classes, propagation, density of states, entanglement.

Bochner Laplacian

Datas:

- (M,g) compact riemannian manifold with $\partial M = \emptyset$
- $L \to M$ hermitian line bundle with a connexion ∇
- $V \in \mathcal{C}^\infty(M,\mathbb{R})$ a potential

The Bochner Laplacian, or Schrödinger operator with magnetic field $\omega = i\,\text{courb}(\nabla)$ is

$$\begin{aligned} \Delta &= \tfrac{1}{2}\nabla^*\nabla + V && \text{acting on } \mathcal{C}^\infty(M,L) \\ &= -\tfrac{1}{2\sqrt{g}}\nabla_i(g^{ij}\sqrt{g}\nabla_j) + V && \text{with } \nabla_i = \partial_{x_i} + \tfrac{1}{i}a_i \\ & && \text{where } d(a_i dx_i) = \omega \end{aligned}$$

Since $\Delta = -\frac{1}{2}g^{ij}\partial_{x_i}\partial_{x_j} +$ lower order derivative terms, Δ is an elliptic differential operator. Hence Δ has a discrete spectrum, bounded from above, eigenvalues with finite multiplicities and smooth eigensections.

Semiclassical limit, $k = \hbar^{-1}$, $k \to \infty$

take $k \in \mathbb{N}$ and replace L by $L^k = L^{\otimes k}$, ∇ by ∇^{L^k} and set

$$\begin{aligned} \hat{H}_k &:= \tfrac{k^{-2}}{2} (\nabla^{L^k})^* \nabla^{L^k} + k^{-1} V \\ &= \tfrac{1}{2} g^{ij} \pi_i \pi_j + b_i k^{-1} \pi_i + k^{-1} V \end{aligned}$$

with $\pi_i = \frac{1}{ik} \partial_{x_i} - a_i$, the dynamical moments, $ik[\pi_i, \pi_j] = \omega_{ij}$.

$\hat{H}_k$ is a semiclassical differential operator with order 0 and symbol $H \in \mathcal{C}^\infty(T^*M, \mathbb{R})$ equal to $H(x, \xi) = \frac{1}{2} g^{ij}(x) \xi_i \xi_j$.

Weyl law

for any $E \in \mathbb{R}$, we have in the large k limit

$$\operatorname{rank} 1_{]-\infty, E]}(\hat{H}_k) = \left(\frac{k}{2\pi} \right)^m (\operatorname{vol}(\{H \leqslant E\}) + o(1))$$

with $m = \dim M$ and vol is for the Liouville form $\Omega^m / m!$ where $\Omega = \sum d\xi_i \wedge dx_i - \omega$.

Landau symbol

Let us consider eigenvalues of $\hat{H}_k$ around $E = 0$ at the scale k^{-1}.
Assume that ω is non-degenerate, so m is even.

Definition of the Landau symbol $L_H(x_0)$ of $\hat{H}_k$ at $x_0 \in M$

- take leading order terms in $\hat{H}_k = \frac{1}{2} g^{ij}\pi_i\pi_j + b_i k^{-1}\pi_i + k^{-1}V$ with the convention $\mathrm{ord}(f) = 0$, $\mathrm{ord}(\pi_i) = \frac{1}{2}\,\mathrm{ord}(k^{-1}) = -1$.
- put $k = 1$ and freeze coordinates at x_0,

Then set

$$L_H(x_0) := \frac{1}{2} g^{ij}(x_0)\pi_i(x_0)\pi_j(x_0) + V(x_0)$$

acting on $L^2(\mathbb{R}^m)$, with $\pi_i(x_0)$ the dynamical moment for $\omega_{ij}(x_0)dx_i \wedge dx_j$.

Surface case

$m = 2$, $g^{ij}(x_0) = \delta_{ij}$, $\omega(x_0) = B(x_0)dx_1 \wedge dx_2$ with $B(x_0) > 0$.
Then the spectrum of $L_H(x_0)$ is $\{B(x_0)(n + \frac{1}{2}) + V(x_0) / n \in \mathbb{N}\}$.

Landau levels for surfaces

set $\lambda_n = B(n+\frac{1}{2}) + V$ so that $\operatorname{spec} L_H(x_0) = \{\lambda_n(x_0) / n \in \mathbb{N}\}$.

Demailly Weyl law (85)

$$\operatorname{rank} 1_{(-\infty,E)}(k\hat{H}_k) = \frac{k}{2\pi}\sum_n \operatorname{vol}(\lambda_n \leqslant E) + o(k)$$

where vol is the volume in M for ω.

n-th Landau level $\mathcal{H}_n$

Assume there exists E_-, E_+ such that $\max \lambda_{n-1} \leqslant E_- \leqslant \min \lambda_n$ and $\max \lambda_n \leqslant E_+ \leqslant \min \lambda_{n+1}$. Set $\mathcal{H}_n = \operatorname{Im} 1_{[E_-,E_+]}(k\hat{H}_k)$.
Then when k is large,

$$\dim \mathcal{H}_n = \frac{k}{2\pi}\operatorname{vol}(M) + (\tfrac{1}{2}+n)\chi(M).$$

Here, $\operatorname{vol}(M)/2\pi$ is the Chern number of L.
For B, V and Gaussian curvature constant, this is known from the 90's. Otherwise this seems to be new.

Comments

1. For $V = 0$ and B constant, Weitzenböck formula $k\hat{H}_k = k\overline{\partial}^*_{L^k}\overline{\partial}_{L^k} + \frac{1}{2}B$ and Kodaira inequalities leads to

$$\mathcal{H}_0 = \{\text{holomorphic sections of } L^k\}$$

the dimension is given by Riemann-Roch theorem.

2. For $V = 0$, B and Gaussian curvature S constant, by Iengo-Li (94), there exists ladder operators $\mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k \otimes K^{-n})$ restricting to isomorphisms between $\mathcal{H}_n$ and lowest Landau level of $k^{-2}\Delta_{L^k \otimes K^{-n}}$. Here K is the canonical bundle. Furthermore, we have a single eigenvalue, $\lambda_n = B(n + \frac{1}{2}) + k^{-1}S\frac{n(n+1)}{2}$.

3. In my proof, the canonical bundle appears through

$$K_x^{-n} = \ker(L_H(x) - \lambda_n(x)).$$

Here, $L_H(x)$ acts on $\mathrm{span}(\pi_{i_1}(x) \dots \pi_{i_n}(x)|1\rangle)$

Higher dimensions

In any dimension m, Demailly's Weyl law holds with the $\lambda_n(x)$ defined as the eigenvalues of $H_L(x)$ counted with multiplicities.

$\mathrm{spec}(L_H(x)) = \{\sum_i B_i(x)(\alpha(i)+\frac{1}{2}) + V(x),\ \alpha \in \mathbb{N}^{m/2}\}$ where $0 < B_1 \leqslant \ldots \leqslant B_{m/2}$ are the g-eigenvalues of ω [1].

Theorem (C 21)

If there exists $E \in \mathbb{R} \setminus \bigcup_x \mathrm{spec}(L_H(x))$, then when k is large,

$$\begin{aligned}\mathrm{rank}\, 1_{]-\infty,E]}(k\hat{H}_k) &= \int_M \mathrm{Ch}(L^k \otimes F)\,\mathrm{Todd}(M) \\ &= \left(\frac{k}{2\pi}\right)^{m/2} \mathrm{vol}(M)\,\mathrm{rank}(F) + \mathcal{O}(k^{\frac{m}{2}-1})\end{aligned}$$

with $F \to M$ the vector bundle with $F_x = \mathrm{Im}\, 1_{]-\infty,E]}(H_L(x))$.

[1] With good coordinates $g_{ij}(x_0) = \delta_{ij}$ and $\omega|_{x_0} = B_1(x_0)dx_1 \wedge dx_2 + \ldots + B_{m/2}(x_0)dx_{m-1} \wedge dx_m$

Assume that $B_1 = \ldots = B_{m/2} = B$ and let $\lambda_n = B(n + \frac{m}{4}) + V$. The $\lambda_n(x)$ are the eigenvalues of $L_H(x)$, $\mathrm{mult}(\lambda_n(x)) = \binom{n+m/2-1}{m/2-1}$.

Let E_-, E_+ be such that $\lambda_{n-1} < E_- < \lambda_n < E_+ < \lambda_{n+1}$ and define the n-th Landau level as $\mathcal{H}_n = \mathrm{Im}(1_{(E_-,E_+)}(k\hat{H}_k))$. Then by the previous theorem, when k is large

$$\dim \mathcal{H}_n = \int_M \mathrm{Ch}(L^k \otimes \mathrm{Sym}^n(T^{1,0}M)) \,\mathrm{Todd}\, M \qquad (1)$$

Earlier results for $B_1 = \ldots = B_{m/2} = 1$ and $V = 0$ so $\mathrm{spec}\, L_H(x) = \frac{m}{4} + \mathbb{N}$:

1. Lowest Landau level ($n = 0$): when ω is Kähler, (1) follows from Riemann-Roch-Hirzebruch theorem and Kodaira vanishing theorem. In the symplectic case, this is a theorem of Guillemin-Uribe (88) and Borthwick-Uribe (96).
2. Higher levels: the existence of gaps was proved by Faure-Tsuji (15)

Dynamics in Landau levels

Assume that $B_1 = \ldots = B_{m/2} = B$ and let $\lambda_n = B(n + \frac{m}{4}) + V$.
Set $\mathcal{H}_n = \operatorname{Im} 1_{[E_-,E_+]}(k\hat{H}_k)$ with $\lambda_{n-1} < E_- < \lambda_n < E_+ < \lambda_{n+1}$.

Let $\Psi \in \mathcal{H}_n$ and define

$$\Psi(t) = \exp(itk\hat{H}_k)\Psi, \qquad t \in \mathbb{R}$$

The L^2-norm of $\Psi(t)$ is $(\int_M |\Psi(t)|^2(x) d\mu_g(x))^{1/2}$, so if $\|\Psi\| = 1$, $|\Psi(t)|^2$ is the probability density function of the particle's position.

Theorem $$|\Psi(kt)|^2 \mu_g = (\Phi_t)_*(|\Psi|^2 \mu_g) + \mathcal{O}(k^{-1})$$

where (Φ_t) is the Hamiltonian flow of λ_n in (M, ω).

More precisely, for any $f \in \mathcal{C}^\infty(M, \mathbb{R})$,
$\int |\Psi(kt)|^2 f \mu_g = \int |\Psi|^2 (f \circ \Phi_t)\, \mu_g + \mathcal{O}_f(k^{-1})$ and the $\mathcal{O}$ is uniform when t remains bounded.

Density of states

Let $(\psi_{k,i})$ be an onb of eigenvectors, $\hat{H}_k \psi_{k,i} = E_{k,i}\psi_{k,i}$.
For any $a, b \in \mathbb{R}$ with $a < b$ and $x \in M$, set

$$N(x, a, b, k) = \sum_{i,\ kE_{k,i} \in [a,b]} |\psi_{k,i}(x)|^2$$

Theorem (C. 21)

if $a, b \notin \operatorname{spec} L_H(x)$, *then*

$$N(x, a, b, k) = \left(\frac{k}{2\pi}\right)^{m/2} \sum_{\ell=0}^{\infty} m_\ell k^{-\ell} + \mathcal{O}(k^{-\infty})$$

with $m_0 = \sharp([a, b] \cap \operatorname{spec} L_H(x))$.

This is proved only for $a, b < E$ with $E \in \mathbb{R} \setminus \bigcup_{x \in M} \operatorname{spec}(L_H(x))$.

Entanglement

Assume that $B_1 = \ldots = B_{m/2} = B$ and let $\lambda_n = B(n + \frac{m}{4}) + V$.
Set $\mathcal{H}_{\leqslant n} := \operatorname{Im} 1_{(-\infty,E)}(k\hat{H}_k)$ with $\lambda_n < E < \lambda_{n+1}$.

Consider the fermion $\Psi = \Psi_{k,1} \wedge \ldots \wedge \Psi_{k,d_k}$ where $(\Psi_{k,i})_{i=1}^{d_k}$ is an onb of $\mathcal{H}_{\leqslant n}$. Its probability in configuration space M^{d_k} is

$$|\det(\Psi_{k,i}(x_j))_{i,j}|^2 d\mu_g^{\otimes d_k}(x_1, \ldots, x_{d_k}).$$

Let A be a domain of M and N_A be the random variable of M^{d_k},

$$N_A(x_1, \ldots, x_{d_k}) = \sharp\{i,\ x_i \in A\}.$$

Theorem

If the boundary of A is smooth, then

$$\mathbb{E}(N_A) \sim C_{n,m} k^{m/2} \operatorname{vol}(A), \qquad \operatorname{var}(N_A) \sim C'_{n,m} k^{(m-1)/2} \operatorname{vol}(\partial A)$$

where the volumes are for the metric Bg.

based on previous work with B. Estienne (18). Related work by Leschke, Sobolev, Spitzer (20).

References

My own work

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