Topological electrostatics

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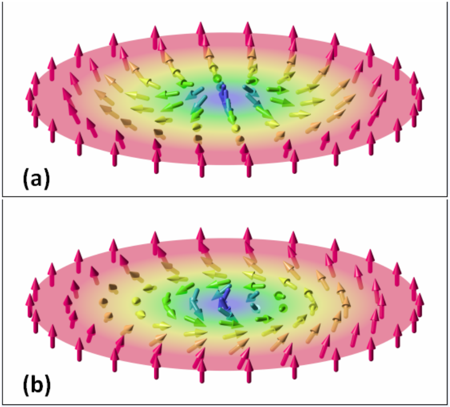
Plan

- Quantum Hall ferromagnets
- Skyrmion lattices in Quantum Hall ferromagnets
- Topological electrostatics

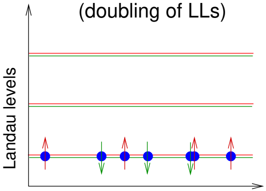
Spin Skyrmions



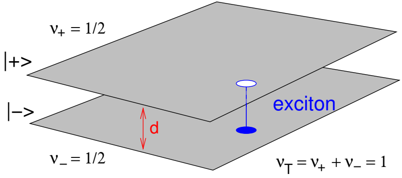
$$\pi_2(\mathrm{S}^2) = \mathbb{Z}$$

(Picture from Wikipedia)

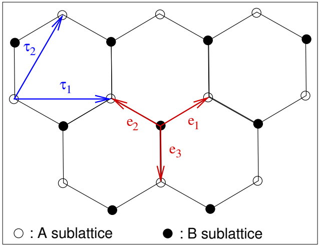
(A) physical spin: SU(2)



(B) bilayer: SU(2) isospin



(C) graphene (2D graphite)



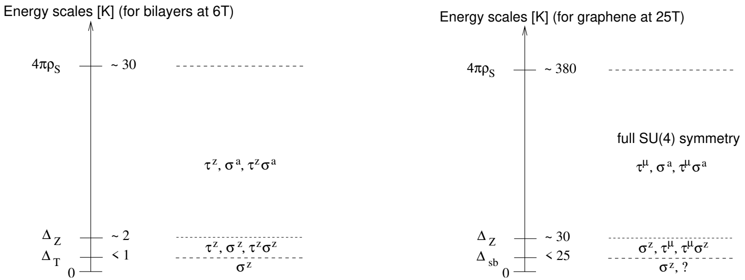
two−fold valley degeneracy
⟶ SU(2) isospin

spin + isospin : SU(4)

Realistic anisotropies

Hamiltonian can approximately have high $SU(4)$ symmetry

- Zeeman anisotropy: $SU(2) \to U(1)$
- Graphene: valley weakly split, $O(a/l_B)$
- Bilayers: charging energy: $SU(2) \to U(1)$; neglect tunnelling



Quantum Hall ferromagnets

N internal states (spin, valley, layer indices, e. g. $N = 4$ for graphene).
Integer filling factor M with $1 \leq M \leq N - 1$.
Large magnetic field $\rightarrow$ Projection onto the lowest Landau level (LLL). Assume that largest sub-leading term is given by Coulomb interactions (small g factor). This selects a ferromagnetic state

Main question: What happens when $\nu = M + \delta\nu$, $\delta\nu << 1$?

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Main question: What happens when $\nu = M + \delta\nu$, $\delta\nu << 1$?

Ferromagnetic state is replaced by slowly varying textures (e. g. Skyrmions lattices for $M = 1$).

Sondhi, Karlhede, Kivelson, Rezayi, PRB **47**, 16419, (1993), Brey, Fertig, Côté and MacDonald, PRL **75**, 2562 (1995)

Skyrmion crystals near $\nu = 1$

Theoretical prediction: Brey, Fertig, Côté and MacDonald, PRL **75**, 2562 (1995)

Specific heat peak: Bayot et al. PRL **76**, 4584 (1996) and PRL **79**, 1718 (1997)

Increase in NMR relaxation: Gervais et al. PRL **94**, 196803 (2005)

Raman spectroscopy: Gallais et al, PRL **100**, 086806 (2008)

Microwave spectroscopy: Han Zhu et al. PRL **104**, 226801 (2010)

LETTERS
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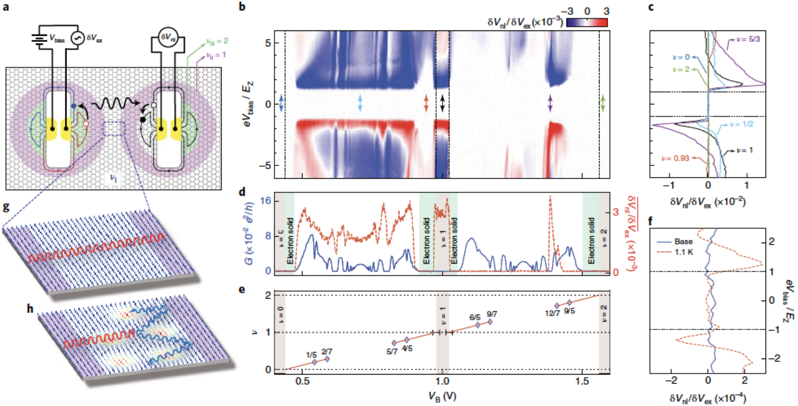
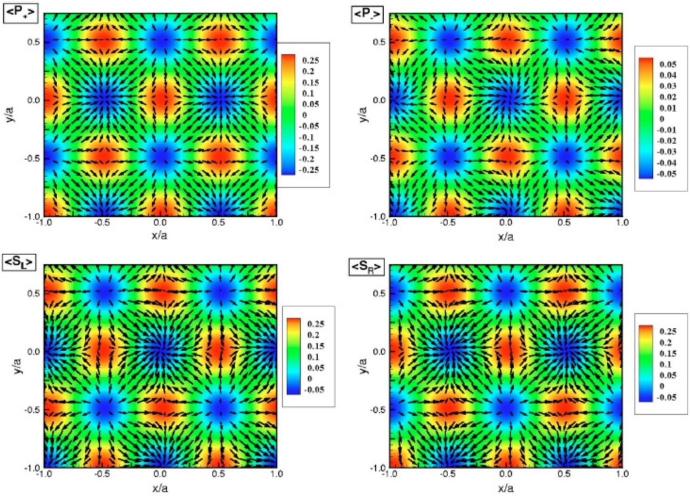
nature physics

Solids of quantum Hall skyrmions in graphene

H. Zhou[1], H. Polshyn[1], T. Taniguchi[2], K. Watanabe[2] and A. F. Young[1*]



Example of entangled textures ($N = 4$, $M = 1$)



Bourassa et al, Phys. Rev. B **74**, 195320 (2006)

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Description of uniform states

Work in lowest Landau level with $\nu = M$, $1 \leq M \leq N-1$. We choose an M-dimensional subspace in $\mathbb{C}^N$, which corresponds to the M occupied internal states. Explicitely, this subspace is generated by the columns of an $N \times M$ matrix V.

Consider now a complete basis $\chi^{(\alpha)}(\mathbf{r})$ in the LLL (orbital degree of freedom). A ferromagnetic state is obtained by taking the Slater determinant $|\mathcal{S}_V\rangle$ built from single particle states of the form $|\Psi^{(i\alpha)}\rangle$, $(1 \leq i \leq M)$, given by:

$$\Psi_a^{(i\alpha)}(\mathbf{r}) = V_{ai}\, \chi^{(\alpha)}(\mathbf{r}), \quad 1 \leq a \leq N$$

Terminology: The continuous set of M-dimensional subspaces in $\mathbb{C}^N$ is a smooth complex manifold of dimension $(N-M)M$, called the Grassmannian $\mathrm{Gr}(M, N)$.

Slater determinants in the LLL associated to smooth textures (I)

Physical space manifold: $\Sigma = \mathbb{R}^2$
Textures: Smooth maps $\Sigma \to \mathcal{M} = \mathrm{Gr}(M, N)$
Explicitely: Pick an $N \times M$ matrix $V_{ij}(\mathbf{r})$ of maps.
This defines a local projector in internal (generalized spin space)
$P_V(\mathbf{r}) = V(\mathbf{r})(V^\dagger(\mathbf{r})V(\mathbf{r}))^{-1}V^\dagger(\mathbf{r})$.
Global symmetry: $V(\mathbf{r}) \to gV(\mathbf{r})$ with $g \in SU(N)$. Spontaneously broken in Skyrmion crystals, giving rise to Goldstone modes, besides phonon modes.
Local gauge symmetry: $V(\mathbf{r}) \to V(\mathbf{r})\Lambda(\mathbf{r})$ with $\Lambda(\mathbf{r}) \in SU(M)$.

Key operation: projection $\mathcal{P}_{LLL}$ onto the lowest Landau level.
The quantum state $|\mathcal{S}_V\rangle$ associated to the classical map is the ground-state of the auxiliary single-particle Hamiltonian:

$$H_{\mathrm{aux},V} = -\mathcal{P}_{LLL}\left(\int d^2\mathbf{r} \sum_{a,b} P_V(\mathbf{r})_{ab}\Psi_a^\dagger(\mathbf{r})\Psi_b(\mathbf{r})\right)\mathcal{P}_{LLL}$$

Slater determinants in the LLL associated to smooth textures (II)

Main effect of $\mathcal{P}_{LLL}$: (Moon et al. (1995), Pasquier (2000),...)

$$\begin{aligned} n_{\mathrm{el}}(\mathbf{r}) &= \frac{M}{2\pi l^2} - Q(\mathbf{r}) + O(l^2) \\ N_{\mathrm{el}} &= M N_\Phi - Q_{\mathrm{top}} \quad \rightarrow \quad \text{CONSTRAINT} \end{aligned}$$

Energy functional:

$$E_{\mathrm{tot}} = E_{\mathrm{loc}} + E_{\mathrm{non-loc}}$$

E_{loc}: exchange energy (generalized ferromagnet), given by a non-linear σ model energy functional.
$E_{\mathrm{non-loc}} = \frac{e^2}{8\pi\epsilon} \int d^2\mathbf{r} \int d^2\mathbf{r}' \frac{Q(\mathbf{r})Q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$.

Slater determinants in the LLL associated to smooth textures (III)

$$E_{\rm tot} = E_{\rm loc} + E_{\rm non-loc}$$

If filling factor is close to M, $E_{\rm non-loc} << E_{\rm loc}$. To find optimal textures, we can therefore:

1. Minimize $E_{\rm loc}$ in the presence of the $N_{\rm el} = MN_{\Phi} - Q_{\rm top}$ constraint. This leads to a continuous family of degenerate configurations, described by holomorphic maps $\Sigma \to Gr(M,N)$.
2. Lift this degeneracy by minimizing $E_{\rm non-loc}$ within this degenerate family. Physically, this favors textures in which the topological charge density is as uniform as possible: this may be described as a problem in *topological electrostatics.*

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Holomorphic maps from the sphere to $\mathbb{CP}(N-1)$ (I)

$S^2 \cong \mathbb{CP}(1) \cong \mathbb{C} \cup \{\infty\}$ so we use one coordinate $z \in \mathbb{C}$.
Kähler potential on the sphere: $\Phi = \frac{1}{\pi} \log(1+|z|^2)$
Volume element: $\omega = \frac{dx \wedge dy}{\pi(1+|z|^2)^2}$
Holomorphic maps $f : S^2 \to \mathbb{CP}(N-1)$: collections of N polynomials $P_1(z), ..., P_N(z)$.
Topological charge: number of intersection points of $f(S^2)$ with an arbitrary hyperplane in $\mathbb{CP}(N-1)$ = maximal degree d of $P_1(z), ..., P_N(z)$.
Topological charge density:

$$Q(z,\bar{z}) = (1+|z|^2)^2 \partial_z \partial_{\bar{z}} \log\left(\sum_{i=1}^{N} |P_i(z)|^2\right)$$

$Q(z,\bar{z})$ is constant when:

$$\sum_{i=1}^{N} |P_i(z)|^2 = (1+|z|^2)^d$$

Holomorphic maps from the sphere to $\mathbb{C}\mathrm{P}(N-1)$ (II)

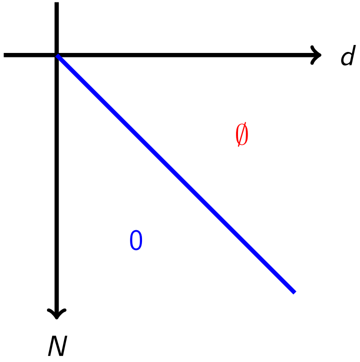
Hermitian scalar product on degree d polynomials:

$$(P,Q)_d = \frac{d+1}{\pi}\int d^2\mathbf{r}\; \frac{\overline{P(z)}Q(z)}{(1+|z|^2)^{d+2}}$$

Orthonormal basis: $e_p(z) = \begin{pmatrix} d \\ p \end{pmatrix}^{1/2} z^p$

General texture of degree d: $P_i(z) = \sum_{i=0}^{d} A_{ij} e_j(z)$
$Q(z,\bar{z})$ is constant when: $A^\dagger A = I_{d+1}$
If $d \geq N$: No solution
If $d \leq N-2$: many solutions, but not all components of the maps are linearly independent.
If $d = N-1$: $AA^\dagger = I_N = A^\dagger A$, so $(P_i, P_j)_d = \delta_{ij}$.
Textures with uniform topological charge density $\Leftrightarrow$ Components form an orthonormal basis.

Holomorphic maps from the sphere to $\mathbb{CP}(N-1)$ (III)



- 0 There exists a unique solution, up to global $SU(N)$ transformations, giving a uniform topological charge density
- ∅ No such solution exists

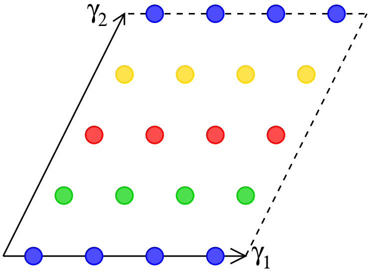
$$\theta(z+\gamma) = e^{a_\gamma z + b_\gamma}\theta(z)$$
$$(\theta,\theta')_d = \int d^2\mathbf{r}\ \exp(-\frac{\pi d|z|^2}{|\gamma_1 \wedge \gamma_2|})\overline{\theta(z)}\theta'(z)$$

Optimal textures ($d = N$)

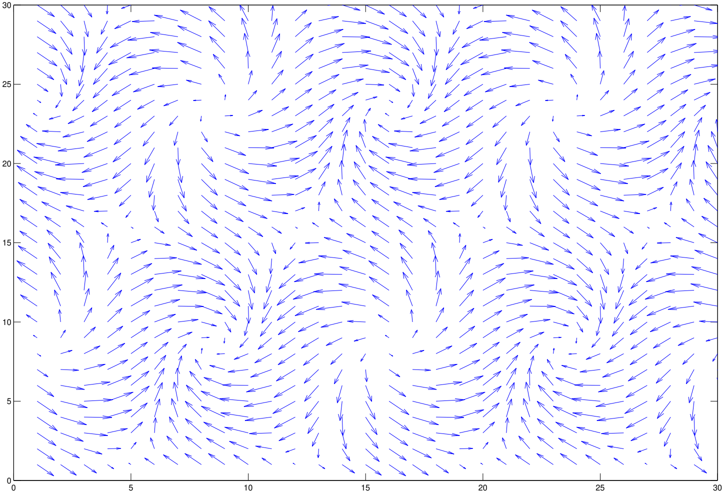
$$|\Psi(z)\rangle = \begin{pmatrix} \theta_0(z) \\ \theta_1(z) \\ . \\ . \\ . \\ \theta_{d-1}(z) \end{pmatrix}$$

$(\theta_i,\theta_j)_d = \delta_{ij}$

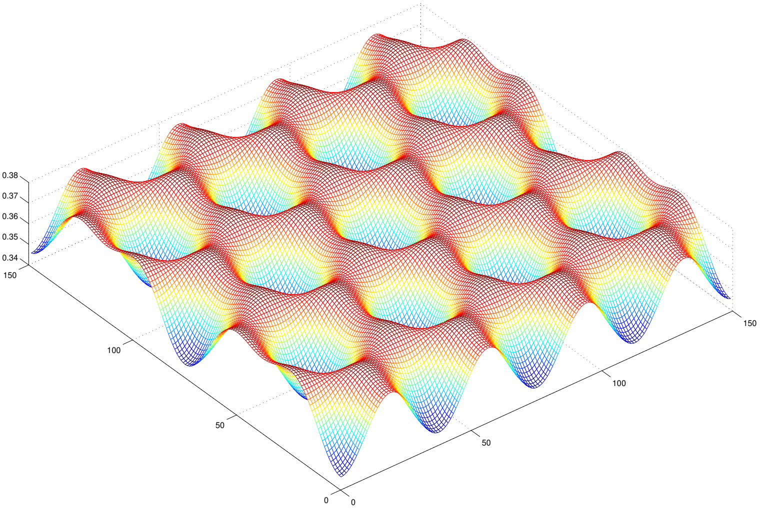
Pattern of zeros (d=4)



$d = N = 2$



$d = N = 4$



Holomorphic maps from the torus to $\mathbb{CP}(N-1)$ (IV)

Spatial variations of topological charge: $Q(r)$ is always γ_1/d and γ_2/d periodic. Unlike on the sphere, $Q(r)$ is not exactly constant.

At large d the modulation contains mostly the lowest harmonic, and its amplitude decays exponentially with d.

Large d behavior for a square lattice:

$$Q(x,y) \simeq \frac{2}{\pi} - 4d e^{-\pi d/2}[\cos(2\sqrt{d}x) - 2e^{-\pi d/2}\cos^2(4\sqrt{d}x) + (x \leftrightarrow y)] + \ldots$$

Only the triangular lattice seems to yield a true local energy minimum. This has been evidenced by computing eigenfrequencies of small deformation modes.

B. Douçot, D. Kovrizhin, R. Moessner, PRL 110, 186802 (2013)

Holomorphic maps from Σ to $\mathbb{CP}(N-1)$ (I)

Components of a map $f : \Sigma \to \mathbb{CP}(N-1)$ were polynomials on the sphere and θ functions on the torus. Note that polynomials have poles at $z \to \infty$, and θ functions are multivalued.

More general construction: Pick a line bundle L over Σ, and choose the components of the maps $s_j(z)$ as global holomorphic sections of L, for $1 \leq j \leq N$.

Recipe for optimal textures: $N =$ dimension of the space of global holomorphic sections of L. Choose components forming an orthonormal basis for a well chosen hermitian product.

Geometric quantization recipe for the hermitian product

ω : volume form associated to constant curvature metric on Σ

h^d : hermitian metric on fibers of L^d whose curvature form equals $-d(2\pi i)\omega$

$$(s, s')_{L,d} = \int_\Sigma h^d(s(x), s'(x))\omega(x)$$

Topological charge form: $\omega_{\mathrm{top}} - \omega = \frac{1}{\pi}\partial_z\partial_{\bar{z}} \log B(z, \bar{z})$.

$B(z, \bar{z})_{L,d} = \sum_{j=1}^{N} h^d(s_j(z), s_j(z))$

For an orthonormal basis $B(z, \bar{z})$ is the Bergman kernel, whose large d asymptotics has been studied a lot in the 90's.

Holomorphic maps from Σ to $\mathbb{CP}(N-1)$ (III)

Bergman kernel asymptotics (Tian, Yau, Zelditch, Catlin, Lu,...(1990 to 2000)):
$B(z,\bar{z}) = d + a_0(z,\bar{z}) + a_{-1}(z,\bar{z})d^{-1} + a_{-2}(z,\bar{z})d^{-2} + ...$, such that $a_j(z,\bar{z})$ is a polynomial in the curvature and its covariant derivatives at $(z,\bar{z})$.

Interesting consequence: If ω is associated to the constant curvature metric on Σ, the previous family of textures have uniform topological charge, up to corrections which are smaller than any power of $1/d$.

"Practical" questions: How to effectively construct such orthonormal bases of sections, when Σ has genus ≥ 2 ?
Optimization of the exponentially small corrections in d with respect to the line bundle L ?

Maps $\Sigma \to \mathrm{Gr}(M, N)$ and rank M vector bundles (I)

We start from a rank M vector bundle $\mathcal{V}$ over Σ, and a choice of N sections $s_i(x)$, $1 \leq i \leq N$ of $\mathcal{V}$, which generate the fiber $\mathcal{V}_x$ at each $x \in \Sigma$.
Using local frames in open subsets U_α covering Σ, each section $s_i(x)$ may be seen as an M-component row-vector. These N rows form an $N \times M$ matrix $V^{(\alpha)}(x)$, and if $x \in U_\alpha \cap U_\beta$:

$$V^{(\alpha)}(x) = V^{(\beta)}(x) t^{(\beta\alpha)}(x)$$

where $t^{(\beta\alpha)}(x)$ are the transition functions of $\mathcal{V}$.
The linear span in $\mathbb{C}^N$ of the columns of $V^{(\alpha)}(x)$ form a well defined $f(x) \in \mathrm{Gr}(M, N)$.
Elements of $\mathcal{V}_x \longleftrightarrow M$-component row-vectors
Elements of $f(x) \longleftrightarrow N$-component column-vectors

$$\mathcal{V}_x \cong f(x)^*$$

Maps $\Sigma \to \mathrm{Gr}(M, N)$ and rank M vector bundles (II)

Basic fact: there exists a 1 to 1 correspondence between:

- Maps $f : \Sigma \to \mathrm{Gr}(M, N)$
- Rank M vector bundles $\mathcal{V}$ over Σ, together with a choice of N sections of $\mathcal{V}$, which generate the fiber $\mathcal{V}_x$ at each $x \in \Sigma$, modulo automorphisms of $\mathcal{V}$.

$$\begin{array}{ccc} \mathcal{V} \cong f^*\mathcal{T}^* & \xrightarrow{\bar{f}} & \mathcal{T}^* \\ s_i \uparrow & 1 \leq i \leq N & \uparrow t_i \\ \Sigma & \xrightarrow{f} & \mathrm{Gr}(M, N) \end{array}$$

$\mathcal{T}^*$: dual of tautological rank M vector bundle over $\mathrm{Gr}(M, N)$.
For $V \in \mathrm{Gr}(M, N)$, $t_i(V)$ is the linear form on V defined by the i-th component in $\mathbb{C}^N$ ($V \subset \mathbb{C}^N$).

Using the Plücker embedding of $\mathrm{Gr}(M, N)$ into $\mathbb{C}P(\tilde{N}-1)$

$$
\begin{array}{ccccc}
\mathrm{Det}\ \mathcal{V} & \xrightarrow{\ \overline{f}\ } & \mathrm{Det}\ \mathcal{T}^* & \xrightarrow{\ \overline{i_{\mathcal{P}}}\ } & \mathcal{O}(1) \\
\uparrow {\scriptstyle s_{i_1} \wedge \ldots \wedge s_{i_M}} & & \uparrow {\scriptstyle t_{i_1} \wedge \ldots \wedge t_{i_M}} & & \uparrow {\scriptstyle x_{i_1,\ldots,i_M}} \\
\Sigma & \xrightarrow{\ f\ } & \mathrm{Gr}(M, N) & \xrightarrow{\ i_{\mathcal{P}}\ } & \mathbb{C}P(\tilde{N}-1)
\end{array}
$$

$\tilde{N} = N!/(M!(N-M)!)$. Suggests to consider $i_{\mathcal{P}}\, f$, which is generated by the $\tilde{N}$ sections $s_{i_1} \wedge \ldots \wedge s_{i_M}$ of Det $\mathcal{V}$.
Main difficulty: An optimal texture $\Sigma \to \mathbb{C}P(\tilde{N}-1)$ is not always of the form $i_{\mathcal{P}}\, f$!

Recipe for optimal $\Sigma \to \mathrm{Gr}(M, N)$ textures

1. Pick a rank M vector bundle $\mathcal{V}$ over Σ
2. Choose N global sections of $\mathcal{V}$, encoded in an $N \times D$ matrix A
3. Apply Plücker's embedding: $i_{\mathcal{P}}\, f$ is described by $\tilde{N}$ sections of Det $\mathcal{V}$, encoded in an $\tilde{N} \times \tilde{D}$ matrix $\tilde{A}$.
4. Try to impose the optimality constraint for projective textures: $\tilde{A}^\dagger \tilde{A} = I_{\tilde{D}}$. Assuming that entries of $\tilde{A}^\dagger \tilde{A}$ are independent functions of those of A, this gives $\tilde{D}^2$ constraints.

$SU(N)$-invariant physical degrees of freedom are given by entries of $A^\dagger A$. This gives D^2 independent $SU(N)$-invariant physical degrees of freedom when $D \leq N$, and $2ND - N^2 < D^2$ degrees of freedom when $D > N$. Optimal textures are described by $D^2 - 2\mathcal{A}(\mathcal{V}) - \tilde{D}^2$ real parameters, where $\mathcal{A}(\mathcal{V})$ is the number of independent (over $\mathbb{C}$) automorphisms of $\mathcal{V}$.

Holomorphic maps from the sphere to $\mathrm{Gr}(M, N)$

- Any rank M vector bundle $\mathcal{V}$ on the sphere decomposes as a direct sum of line bundles $\mathcal{V} = \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) ... \oplus \mathcal{O}(d_M)$, with $d_1 + d_2 + ... + d_M = d$ is the total topological charge (Grothendieck 1957).
- Sections of $\mathcal{O}(d)$ on S^2 form a complex space of dimension $d + 1$ (spin $S = d/2$ representation of $SU(2)$), realized by polynomials in z with maximal degree d.
- From this, we get that $D = d + M$ and $\tilde{D} = d + 1$
- For $M = 2$, $\mathcal{A}(\mathcal{V}) = d_2 - d_1 + 2$ if $d_1 < d_2$, and $\mathcal{A}(\mathcal{V}) = 3$ if $d_1 = d_2$. If we choose d_1 and d_2 to minimize $\mathcal{A}(\mathcal{V})$ at fixed d, we predict $2d - 3$ $SU(N)$-invariant free parameters for constant topological charge textures (if $N \geq D = d + 2$).

$SU(N)$-invariant deformation modes on the sphere

Assumptions

- $\tilde{D}^2$ constraints are independent
- Automorphisms of $\mathcal{V}$ generate $\mathcal{A}(\mathcal{V})$ independent small deformations of $A^\dagger A$

$N \backslash d$	1	2	3	4	5	6	7	8	9	10
3	0	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	0	1	2	1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	0	1	3	4	3	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	0	1	3	5	6	5	2	$\emptyset$	$\emptyset$	$\emptyset$
7	0	1	3	5	7	8	7	4	$\emptyset$	$\emptyset$
8	0	1	3	5	7	9	10	9	6	1
9	0	1	3	5	7	9	11	12	11	8
10	0	1	3	5	7	9	11	13	14	13

Holomorphic maps from the torus to $\mathrm{Gr}(M, N)$

- There exist undecomposable vector bundles on a torus, which have been completely classified by Atiyah (1957).
- Their spaces of sections can be explicitely described in terms of θ functions and their derivatives, (Polishchuk & Zaslow, 1998).
- $D = \tilde{D} = d$ (Riemann-Roch).
- Some undecomposable bundles have no automorphisms ($\mathcal{A}(\mathcal{V}) = 0$). Then, $A^\dagger A = I_d$ is a solution, and seems to be the only one. This happens for example if M and d are relatively prime. We therefore recover a situation very similar to $M = 1$.
- For some other undecomposable bundles, a simple counting argument suggests that in general, there are no solutions to $\tilde{A}^\dagger \tilde{A} = I_d$. However, for $M = 2$ and d even, D. Kovrizhin has found special solutions to $\tilde{A}^\dagger \tilde{A} = I_d$.

Final remarks

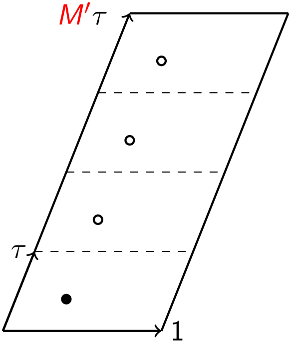
- Qualitative differences between $M = 1$ and $M \geq 2$. Main manifestation: existence of new $SU(N)$-invariant deformation modes on the sphere. Their physical interpretation remains unclear.
- On the torus, special role played by vector bundles without automorphisms. Related to the concept of stability, which plays a crucial role in many situations (existence of *Hermitian-Einstein metrics*, construction of moduli spaces,...)
- Other situations where getting a uniform topological density is useful: Chern band insulators, generation of artificial gauge fields in cold atom systems, optimization of quantum topological pumps...

Undecomposable holomorphic vector bundles on a torus

Wanted: Undecomposable vector bundle $\mathcal{V}$ of rank M and degree d on the $(1,\tau)$-torus. With $h = \gcd(M, d)$, we write $(M, d) = h(M', d')$. Then (Atiyah, 1957):

$$\mathcal{V} = \pi_*^{(M')}(L' \otimes F_h)$$

L': a line bundle of degree d' over the $(1, M'\tau)$-torus.
F_h: an undecomposable bundle of rank h and degree 0 over the $(1, M'\tau)$-torus.
$\pi_*^{(M')}$: projection from the $(1, M'\tau)$-torus onto the $(1,\tau)$-torus.
Stable bundles correspond to $h = 1$.



Kähler manifolds

$\mathcal{M}$ complex manifold with local complex coordinates w_i.
$\mathcal{M}$ is equipped with an Hermitian metric

$$ds^2 = \sum_{ij} h_{ij}\, dw_i d\bar{w}_j$$

such that the corresponding associated (1,1) form

$$\omega = \frac{i}{2} \sum_{ij} h_{ij}\, dw_i \wedge d\bar{w}_j$$

is closed.
This implies that, locally, the metric derives from a Kähler potential Φ, i.e. that:

$$h_{ij} = \frac{\partial^2 \Phi}{\partial w_i \partial \bar{w}_j}$$

Grassmannians are Kähler manifolds

Choice of local coordinates on $\mathrm{Gr}(M, N)$: Pick a rank M, $N \times M$ matrix V. Then it has at least one non-zero $M \times M$ minor determinant. Assuming this is the first one, we get a dense open subset of $\mathrm{Gr}(M, N)$. $V = \begin{pmatrix} V_u \\ V_d \end{pmatrix}$. Multiplying V on the right by V_u^{-1} leads to the same M-dimensional subspace. This changes V into

$$\begin{pmatrix} I_m \\ W \end{pmatrix}$$

where $W = V_d V_u^{-1}$ is an arbitrary $(N - M) \times M$ matrix.

Kähler potential:

$$\Phi(W, W^\dagger) = \frac{1}{\pi} \log \det(I + W^\dagger W)$$

Energy functionals for maps to Kähler manifolds

Classical energy functional for a map $(x, y) \to (w_i)$:

$$E = \frac{g}{2} \int d^2\mathbf{r}\, h_{ij}(w(\mathbf{r}), \bar{w}(\mathbf{r})) \nabla w_i . \nabla \bar{w}_j$$

$$E = g \int d^2\mathbf{r}\, h_{ij}(\partial_z w_i \partial_{\bar{z}} \bar{w}_j + \partial_{\bar{z}} w_i \partial_z \bar{w}_j)$$

The topological charge density is defined by:

$$Q = \int d^2\mathbf{r}\, f^* \omega$$

Explicitely:

$$Q = \int d^2\mathbf{r}\, h_{ij}(\partial_z w_i \partial_{\bar{z}} \bar{w}_j - \partial_{\bar{z}} w_i \partial_z \bar{w}_j)$$

$d\omega = 0$ implies that Q does not change to first order under any infinitesimal variation of the map f, so Q depends only on the homotopy class of f. In many interesting situations, Q takes only integer values.

Bogomolnyi inequality and its consequences

$$E = g(A+B)$$
$$Q = A - B$$

$$A = \int d^2\mathbf{r}\; h_{ij}\partial_z w_i \partial_{\bar{z}} \bar{w}_j \quad B = \int d^2\mathbf{r}\; h_{ij}\partial_{\bar{z}} w_i \partial_z \bar{w}_j$$

Since $h_{ij} = \bar{h}_{ji}$ is positive definite, A and B are both real and non-negative. Then $A + B \geq |A - B|$, so:

$$E \geq g|Q|$$

Minimal energy configurations with fixed Q:
If $Q > 0$, $B = 0$, so $\partial_{\bar{z}} w_i = 0$: minimal configurations are holomorphic.
If $Q < 0$, $A = 0$, so $\partial_z w_i = 0$: minimal configurations are anti-holomorphic.