K-theoretical Bulk boundary correspondence for systems whose spectrum is not bounded from below

Johannes Kellendonk

Université Claude Bernard Lyon 1

j.w. Tom Stoiber Universität Erlangen-Nürnberg

The bulk-boundary correspondence (BBC) relates topol. invariants of the bulk (interior) of a material to topol. invariants localised at its boundary. For the IQHE

$$\sigma_{Hall} = \sigma_{edge}$$

Hatsugai (1996) proposed a version of BBC for periodic models (complex analysis).

In joint work with Schulz-Baldes & Richter we proposed a purely topological version using K-theory of C^*-algebras and cyclic cohomology for 2-d magnetic operators

- for tight binding models (bounded spectrum) in 2003
- for continuous models (differential operators, bounded below) in 2004

Use of C^*-algebras allows to treat aperiodic systems (Bellissard) and to treat disorder.

Elbau & Graf (2002) gave an analytic proof of BBC.

The book by Prodan & Schulz-Baldes "K-theoretic Bulk-Boundary Correspondence .." (Springer, 2016) develops the theory for tight binding models in full detail.

Since recently, Graf, Jud, Tauber speak of violation of BBC in specific periodic models (shallow water, massive Dirac) which have spectrum not bounded from below.

Which problems arise with for systems not bounded below?

Principles of the K-theoretic BBC in 3 steps

We work throughout in the 1-particle 0-temperature approx. (free fermions), no FQHE!

1. Topological setup. Create a correspondence between bulk and edge through an extension of C^*-algebras

$$\mathcal{E} \hookrightarrow \hat{A} \twoheadrightarrow \mathcal{A}$$

- $\mathcal{A}$ = algebra of observables in the bulk (no edge) (defined by phys. principles)
- $\mathcal{E}$ = algebra of observables localized near the edge
- $\hat{A}$ "links $\mathcal{A}$ and $\mathcal{E}$ topologically" at best with trivial K-theory

Then there is a boundary morphism (abstract BBC)

$$\exp : K_0(\mathcal{A}) \to K_1(\mathcal{E})$$

Principles of the K-theoretic BBC in 3 steps

We work throughout in the 1-particle 0-temperature approx. (free fermions), no FQHE!

1. Topological setup. Create a correspondence between bulk and edge through an extension of C^*-algebras

$$\mathcal{E} \hookrightarrow \hat{\mathcal{A}} \twoheadrightarrow \mathcal{A}$$

- $\mathcal{A}$ = algebra of observables in the bulk (no edge) (defined by phys. principles)
- $\mathcal{E}$ = algebra of observables localized near the edge
- $\hat{A}$ "links $\mathcal{A}$ and $\mathcal{E}$ topologically" at best with trivial K-theory

Then there is a boundary morphism (abstract BBC)

$$\exp : K_0(\mathcal{A}) \to K_1(\mathcal{E})$$

2. Affiliation problem. Relate the bulk Hamiltonian H to $\mathcal{A}$

- If H is bounded then $H \in A$. But if H is unbounded?
- Goal: The topological phase of a gapped H is a K-theory class of $\mathcal{A}$.

Principles of the K-theoretic BBC in 3 steps

We work throughout in the 1-particle 0-temperature approx. (free fermions), no FQHE!

1. Topological setup. Create a correspondence between bulk and edge through an extension of C^*-algebras

$$\mathcal{E} \hookrightarrow \hat{\mathcal{A}} \twoheadrightarrow \mathcal{A}$$

- $\mathcal{A}$ = algebra of observables in the bulk (no edge) (defined by phys. principles)
- $\mathcal{E}$ = algebra of observables localized near the edge
- $\hat{A}$ "links $\mathcal{A}$ and $\mathcal{E}$ topologically" at best with trivial K-theory

Then there is a boundary morphism (abstract BBC)

$$\exp : K_0(\mathcal{A}) \to K_1(\mathcal{E})$$

2. Affiliation problem. Relate the bulk Hamiltonian H to $\mathcal{A}$
   - If H is bounded then $H \in A$. But if H is unbounded?
   - Goal: The topological phase of a gapped H is a K-theory class of $\mathcal{A}$.
3. Numerical BBC. Bulk-boundary correspondence between numerical invariants.
   - Need additive functionals from K-groups to $\mathbb{R}$ (measure topological phases).
   - Dual theory: K-homology with duality pairing given by indices ($\mathbb{Z}$-valued (or $\mathbb{Z}/2$))
   - "Dual" theory cyclic cohomology with Connes pairing (Chern-numbers, $\mathbb{C}$-valued).

Topological setup I

The observable algebra of the bulk is faithfully represented on $L^2(\mathbb{R}^d)$

$$\mathcal{A} := \langle V(x)f(D) | V \in \mathcal{C}, f \in C_0(\mathbb{R}^d)\rangle_{C^*}$$

$\mathcal{C} \in C(\mathbb{R}^d)$ alg. of bounded potentials describing the long-range order structure of the material (i.e. $\mathcal{C}$ = periodic functions for a crystal, $\mathcal{C} = \mathbb{C}$ for constant potential)

D = momentum operator, possibly coupled to a vector pot. for magn. field B.
$C_0(\mathbb{R}^d)$ = continuous function vanishing at infinity.

Operators of $\mathcal{A}$ have integral kernels with coeffs. in $\mathcal{C}$.

$$\mathcal{A} \cong \mathcal{C} \rtimes_{\alpha,B} \mathbb{R}^d$$

Topological setup I

The observable algebra of the bulk is faithfully represented on $L^2(\mathbb{R}^d)$

$$\mathcal{A} := \langle V(x)f(D) | V \in \mathcal{C}, f \in C_0(\mathbb{R}^d) \rangle_{C^*}$$

$\mathcal{C} \in C(\mathbb{R}^d)$ alg. of bounded potentials describing the long-range order structure of the material (i.e. $\mathcal{C}$ = periodic functions for a crystal, $\mathcal{C} = \mathbb{C}$ for constant potential)

D = momentum operator, possibly coupled to a vector pot. for magn. field B.
$C_0(\mathbb{R}^d)$ = continuous function vanishing at infinity.

Operators of $\mathcal{A}$ have integral kernels with coeffs. in $\mathcal{C}$.

$$\mathcal{A} \cong \mathcal{C} \rtimes_{\alpha,B} \mathbb{R}^d$$

Introduce a boundary at $s \in \mathbb{R}$. Consider operators on $L^2(\mathbb{R}^d \times \mathbb{R})$

$$\mathcal{E} := \langle V(x)f(D)\phi(x_\perp - s) | V \in \mathcal{C}, f \in C_0(\mathbb{R}^d), \phi \in C_0(\mathbb{R}) \rangle_{C^*}$$

Operators of $\mathcal{E}$ have integral kernels with coeffs. in $\mathcal{C} \otimes C_0(\mathbb{R})$ (decay away from bdry)

$$\mathcal{E} \cong \mathcal{A} \rtimes_{\hat{\alpha}_\perp} \mathbb{R} \cong \mathcal{C} \rtimes_{\alpha_{||},B} \mathbb{R}^{d-1} \otimes \mathcal{K}(L^2(\mathbb{R}))$$

Topological setup I

The observable algebra of the bulk is faithfully represented on $L^2(\mathbb{R}^d)$

$$\mathcal{A} := \langle V(x)f(D) | V \in \mathcal{C}, f \in C_0(\mathbb{R}^d)\rangle_{C^*}$$

$\mathcal{C} \in C(\mathbb{R}^d)$ alg. of bounded potentials describing the long-range order structure of the material (i.e. $\mathcal{C}$ = periodic functions for a crystal, $\mathcal{C} = \mathbb{C}$ for constant potential)

D = momentum operator, possibly coupled to a vector pot. for magn. field B.
$C_0(\mathbb{R}^d)$ = continuous function vanishing at infinity.

Operators of $\mathcal{A}$ have integral kernels with coeffs. in $\mathcal{C}$.

$$\mathcal{A} \cong \mathcal{C} \rtimes_{\alpha,B} \mathbb{R}^d$$

Introduce a boundary at $s \in \mathbb{R}$. Consider operators on $L^2(\mathbb{R}^d \times \mathbb{R})$

$$\mathcal{E} := \langle V(x)f(D)\phi(x_\perp - s) | V \in \mathcal{C}, f \in C_0(\mathbb{R}^d), \phi \in C_0(\mathbb{R})\rangle_{C^*}$$

Operators of $\mathcal{E}$ have integral kernels with coeffs. in $\mathcal{C} \otimes C_0(\mathbb{R})$ (decay away from bdry)

$$\mathcal{E} \cong \mathcal{A} \rtimes_{\hat{\alpha}_\perp} \mathbb{R} \cong \mathcal{C} \rtimes_{\alpha_{||},B} \mathbb{R}^{d-1} \otimes \mathcal{K}(L^2(\mathbb{R}))$$

Link these algebras through operators on $L^2(\mathbb{R}^d \times \mathbb{R})$

$$\hat{A} := \langle V(x)f(D)\phi(x_\perp - s) | V \in \mathcal{C}, f \in C_0(\mathbb{R}^d), \phi \in C_{0,*}(\mathbb{R})\rangle_{C^*}$$

ϕ vanishing at $-\infty$ having a limit at $+\infty$ (Wiener-Hopf extension).

$$\mathcal{E} \hookrightarrow \hat{\mathcal{A}} \overset{s \to -\infty}{\twoheadrightarrow} \mathcal{A}$$

Let A be a unital C^*-algebra.

$$GL(A)^{s.a.} = \{h \in A : h^{-1} \in A, h \text{ self adjoint}\}$$

$K_0(A)$ is the stabilised, grothendiecked version of $GL(A)^{s.a.}/\sim_{homotopy}$.

$$M_n(A) \ni h \mapsto \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \in M_{n+1}(A)$$

$V(A) = \bigcup_n GL(M_n(A)^{s.a.}/\sim_{homotopy}$, abelian semigroup under

$$[h_1] + [h_2] = \left[\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\right]$$

$K_0(A)$ is the group of formal differences of elements of $V(A)$.

$K_1(A)$ is the stabilised version of $GL(A)/\sim_{homotopy}$

If A is not unital then we add a unit $A^\sim = A + \mathbb{C}1$.

$K_i(A)$ is the subgroup of classes from $K_i(A^\sim)$ not coming from the unit.

The boundary map

Theorem (Fundamental theorem of K-theory.)

Let $\mathcal{E} \overset{i}{\hookrightarrow} \hat{\mathcal{A}} \overset{q}{\twoheadrightarrow} \mathcal{A}$. Then

$$\begin{array}{ccccc} K_0(\mathcal{E}) & \rightarrow & K_0(\hat{\mathcal{A}}) & \rightarrow & K_0(\mathcal{A}) \\ \uparrow \text{ind} & & & & \downarrow \exp \\ K_1(\mathcal{A}) & \leftarrow & K_1(\hat{\mathcal{A}}) & \leftarrow & K_1(\mathcal{E}) \end{array}$$

is exact. If $K_i(\hat{\mathcal{A}}) = 0$ then $\exp$ is an isomorphism.

$\exp$ is induced by $\exp : GL(\mathcal{A}^{\sim})^{s.a.}/\sim_{hom} \rightarrow GL(\mathcal{E}^{\sim})/\sim_{hom}$

$$\exp([h]) := [\exp(\imath\pi(\chi(\hat{h}) + 1))]$$

- $\hat{h} \in \hat{\mathcal{A}}$ a lift of h, i.e. $q(\hat{h}) = h$.
- h has gap Δ at 0. $\chi : \mathbb{R} \rightarrow [-1, 1]$ is continuous constant $= -1$ to the left, constant $= 1$ to the right of Δ.

$[\exp(\imath\pi(\chi(\hat{h}) + 1))]$ does not depend on the choice of χ.

In the continuous model (differential operators) Hamiltonians are unbounded. Elements of $\mathcal{A}$ are bounded.

Let A be faithfully (and non-degenerately) represented on $\mathcal{H}$.

The multiplier algebra $\mathcal{M}(A)$ of A is the norm-closed C^*-algebra of $\mathcal{B}(\mathcal{H})$ given by operators $T \in \mathcal{B}(\mathcal{H})$ s.th. $Ta, aT \in A$ for all $a \in A$.

1. $\mathcal{M}(A)$ is unital and contains A as an ideal.
2. If A is stable ($A \otimes \mathcal{K} \cong A$) then $K_i(\mathcal{M}(A)) = \{0\}$. (Our algebras above $\mathcal{A}$ and $\mathcal{E}$ are typically stable.)

Definition

An operator T is affiliated to A if its bounded transform $F(T) := T(1 + T^*T)^{-\frac{1}{2}}$ belongs to $\mathcal{M}(A)$ and $(1 + T^*T)^{-\frac{1}{2}}A$ is dense in A.

If T is invertible and self adjoint then $F(T)$ is homotopic to the spectral flattening of T.

Affiliation

Definition

Let T be affiliated to $M_n(A)$. $(F(T) = T(1 + T^*T)^{-\frac{1}{2}} \in \mathcal{M}(M_n(A)))$

- We say that T is strongly affiliated to $\mathcal{A}$ if its bounded transform

$$F(T) \in M_n(A^\sim)$$

- We say that T is resolvent affiliated to $\mathcal{A}$ if its resolvents

$$(T + z)^{-1} \in M_n(A^\sim), \forall z \in \rho(T)$$

Lemma

Let H be self adjoint affiliated to A.

- *If H is resolvent affiliated and bounded from below then it is strongly affiliated.*
- *Strong affiliation is preserved under infinitesimal H-bounded perturbations.*

Example

Consider the massive 2d Dirac operator (in Fourier space)

$$H(k) = \begin{pmatrix} m & ik_1 + k_2 \\ ik_1 - k_2 & -m \end{pmatrix}$$

Its bounded transform

$$F(H(k)) = (1 + m^2 + k_1^2 + k_2^2)^{-\frac{1}{2}} \begin{pmatrix} m & ik_1 + k_2 \\ ik_1 - k_2 & -m \end{pmatrix}$$

is a multiplier of $M_2(A)$ with $\mathcal{A} = \mathbb{C} \rtimes_\alpha \mathbb{R}^2 \overset{Fourier}{\cong} C_0(\mathbb{R}^2)$.

It is not strongly affiliated as $F(H(k))$ does not converge to a constant matrix as $|k| \to \infty$.
But it is resolvent affiliated to $\mathcal{A}$, because

$$\begin{pmatrix} m & ik_1 + k_2 \\ ik_1 - k_2 & -m \end{pmatrix}^{-1} \xrightarrow{|k|\to\infty} 0$$

Example

Consider the massive 2d Dirac operator (in Fourier space)

$$H(k) = \begin{pmatrix} m & ik_1 + k_2 \\ ik_1 - k_2 & -m \end{pmatrix}$$

Its bounded transform

$$F(H(k)) = (1 + m^2 + k_1^2 + k_2^2)^{-\frac{1}{2}} \begin{pmatrix} m & ik_1 + k_2 \\ ik_1 - k_2 & -m \end{pmatrix}$$

is a multiplier of $M_2(A)$ with $\mathcal{A} = \mathbb{C} \rtimes_\alpha \mathbb{R}^2 \overset{Fourier}{\cong} C_0(\mathbb{R}^2)$.

It is not strongly affiliated as $F(H(k))$ does not converge to a constant matrix as $|k| \to \infty$.
But it is resolvent affiliated to $\mathcal{A}$, because

$$\begin{pmatrix} m & ik_1 + k_2 \\ ik_1 - k_2 & -m \end{pmatrix}^{-1} \overset{|k|\to\infty}{\longrightarrow} 0$$

On the other hand, the regularised massive 2d Dirac operator

$$H_\epsilon(k) = \begin{pmatrix} m + \epsilon k^2 & ik_1 + k_2 \\ ik_1 - k_2 & -m - \epsilon k^2 \end{pmatrix}$$

is strongly affiliated as

$$F(H_\epsilon(k)) \overset{|k|\to\infty}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathcal{A}^\sim)$$

Suppose H is self adjoint, invertible (with gap Δ) and strongly affiliated. Then $F(H) \in Gl^{s.a.}(M_n(A^\sim))$ and thus defines an element of $K_0(A)$, the bulk invariant.

We therefore can apply the boundary map of K-theory to obtain a boundary invariant.

- Look for a lift $\hat{F} \in \hat{A}^\sim$ of $F(H)$, i.e. an element such that $\hat{F} \stackrel{s\to-\infty}{\longrightarrow} F(H) \in A^\sim$.
- Choose $\chi : \mathbb{R} \to [-1,1]$, continuous constant $= -1$ to the left, constant $= 1$ to the right of Δ.
- The $K_1(\mathcal{E})$-class of $\exp(\imath\pi(\chi(\hat{F}) + 1))$ is the boundary invariant.

How to construct a lift?

In the work of 2004, H was given as a differential operator, resolvent affiliated and bounded from below (magnetic Laplacian with homogeneous bounded potential). There we lifted $F(H)$ as follows:

- Restrict H to the halfspace $\{(x_\parallel, x_\perp) \in \mathbb{R}^d : x_\perp \geq s\}$ and employ Dirichlet boundary conditions. Call that $\hat{H}_s$.
- Take the bounded transform of $\hat{H}_s$ to define $\hat{F} := F(\hat{H}_s)$.

There is a lot of flexibility to express the boundary class. Essentially, we only need a non-vanishing continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ which is 1 at $\pm, \infty$ winds once around 0 and this at least partly when varying over the gap. Then

$$\exp(\imath\pi(\chi(\hat{F}) + 1) \sim_h \varphi(\hat{H}_s)$$

With an appropriate choice $\varphi(\hat{H}_s)$ is the unitary implementing time evolution of the states in the gap (edge states) by time $t = \frac{2\pi}{|\Delta|}$.
Questions:

1. Can we always construct a lift like this?
2. Can we take other boundary conditions to construct a lift?

Answer to 2: No! See talk by G.M. Graf.

If H is not strongly affiliated so that $F(H) \in \mathcal{M}(A) \backslash \mathcal{A}^{\sim}$ we need to work with multipliers. Simply replacing $\mathcal{A}$ by $\mathcal{M}(A)$ is not good enough as $K_i(\mathcal{M}(A)) = 0$.

Idea: Work with pairs which differ not too much.

Let

$$M^{(2)}(A) := \{(m_1, m_2) \in \mathcal{M}(A) \oplus \mathcal{M}(A) : m_2 - m_1 \in A\}$$

Lemma

If A is stable then the inclusion $A \stackrel{\imath}{\hookrightarrow} M^{(2)}(A)$

$$\imath(a) = (0, a)$$

induces an isomorphism on K-theory $K_i(A) \stackrel{\imath_*}{\cong} K_i(M^{(2)}(A))$.

$$\begin{array}{ccccc} \mathcal{E} & \hookrightarrow & \hat{\mathcal{A}} & \twoheadrightarrow & \mathcal{A} \\ \downarrow \imath & & \downarrow \imath & & \downarrow \imath \\ M^{(2)}(\mathcal{E}) & \hookrightarrow & M^{(2)}(\hat{\mathcal{A}}) & \twoheadrightarrow & M^{(2)}(\mathcal{A}) \end{array}$$

Hence the six-term exact sequence obtained from $\mathcal{E} \hookrightarrow \hat{\mathcal{A}} \twoheadrightarrow \mathcal{A}$ is isomorphic to the one from $M^{(2)}(\mathcal{E}) \hookrightarrow M^{(2)}(\hat{\mathcal{A}}) \twoheadrightarrow M^{(2)}(\mathcal{A})$.

relative BBC for relatively strongly affiliated operators

Definition

A pair (T_1, T_2) of operators affiliated to A is relatively strong affiliated to A if $F(T_1) - F(T_2) \in A$. In other words $(T_1, T_2) \in M^{(2)}(A)$.

The relative BBC for a pair (H_1, H_2) of relatively strongly affiliated Hamiltonians with common gap is now given by the exponential map of the six term exact sequence associated to $M^{(2)}(\mathcal{E}) \hookrightarrow M^{(2)}(\hat{\mathcal{A}}) \twoheadrightarrow M^{(2)}(\mathcal{A})$.

Question: In the case of differential operators, can we construct lifts through boundary conditions?

Cyclic co-cycles define functionals on K-theory via Connes pairing.

Suppose we have a densely defined faithful lower-semicontinuous trace $\mathcal{T} : A \to \mathbb{C}$ and an action α of $\mathbb{R}^n$ on A which leaves the trace invariant. If the $\mathbb{R}^n$-action is sufficiently regular then $\delta_i(a) = \lim_{t\to 0} t^{-1}(\alpha_{te_i}(a) - a)$ yields a collection of n commuting densely defined derivations on A and we get a densely defined multilinear map on $A^{\otimes(n+1)} \to \mathbb{C}$

$$ch_{\mathcal{T},\alpha}(a_0, \cdots, a_n) = c_n \mathcal{T}(a_0 \delta_{[1,} a_1 \cdots \delta_{,n]} a_n)$$

which yields a well defined map on $GL^{s.a}(A)/\sim$ *homotopy*

$$\langle ch_{\mathcal{T},\alpha}, [h]\rangle = ch_{\mathcal{T},\alpha}(h, \cdots, h)$$

(need spectrally flattened smooth representative h) and on $GL(A)/\sim$ *homotopy*

$$\langle ch_{\mathcal{T},\alpha}, [h]\rangle = ch_{\mathcal{T},\alpha}(h^{-1}, h, \cdots h^{-1}, h)$$

Then, in the context of the Wiener-Hopf extension

$$\langle ch_{\mathcal{T},\alpha}, [h]\rangle = \langle ch_{\hat{\mathcal{T}},\alpha\times\hat{\alpha}_\perp}, \exp[h]\rangle = \langle ch_{\hat{\mathcal{T}},\alpha_\parallel}, \exp[h]\rangle$$

(the latter if α involves translation $\perp$ to the boundary).
This is the numerical BBC.

Consider a 2-dim strongly affiliated Hamiltonian H, periodic along the boundary. Let $n = 2$.

Suppose we can lift H using a boundary condition, getting $\hat{H}_s$.

Then the l.h.s. $\langle ch_{\mathcal{T},\alpha}, [H]\rangle$ is the Chern number of the Fermi projection and the r.h.s. $\langle ch_{\hat{\mathcal{T}},\alpha_{||}}, \exp[H]\rangle$ the spectral flow of $\hat{H}_s(k)$ through a fiducial line in the gap when $k_{||}$ varies over one period.

This cannot be violated. If the two quantities are not equal then the boundary conditions employed do not yield a lift, i.e. $F(\hat{H}_s)$ is not strongly affiliated to $\hat{\mathcal{A}}$.

In principle the numerical BBC for strongly affiliated operators leads directly to a numerical relative BBC by combining $ch_{\mathcal{T},\alpha}$ with $\imath : \mathcal{A} \to M^{(2)}(A)$,

$$\langle ch_{\mathcal{T},\alpha}^{M^{(2)}(A)}, [(h_1, h_2)]\rangle := \langle ch_{\mathcal{T},\alpha}^{A}, \imath_*^{-1}[(h_1, h_2)]\rangle$$

But it might be difficult to compute $\imath_*^{-1}[(h_1, h_2)]$. If (h_1, h_2) have more regularity, then this chern number can be obtained as a difference

$$\langle ch_{\mathcal{T},\alpha}^{M^{(2)}(A)}, [(h_1, h_2)]\rangle = c_n \mathcal{T}(h_2 \delta_{[1,} h_2 \cdots \delta_{,n]} h_2 - h_1 \delta_{[1,} h_1 \cdots \delta_{,n]} h_1)$$

The regularity needed for h_1 and h_2 is that $t \mapsto \alpha_{te_i}(h_1)$ is smooth in the norm topology and $t \mapsto \alpha_{te_i}(h_2 - h_1)$ smooth in the $\|\cdot\| + \|\cdot\|_{\mathcal{T}}$ topology.