Purely linear response of the quantum Hall current to space-adiabatic perturbations

Giovanna Marcelli

joint work arXiv:2112.03071 with D. Monaco

SISSA

Quantum Hall effect and Topological Phases - Strasbourg - June 20, 2022

Quantum Hall effect (QHE):

1. QHE as bulk effect
   - E_2 induced by a voltage difference
   - $j_1 = \sigma_B E_2$
   - $j_1 = \sigma_B E_2 + O(E_2{}^\infty)$

2. QHE as a pump effect
   - E_1 induced by a change in the flux $\phi(s)$
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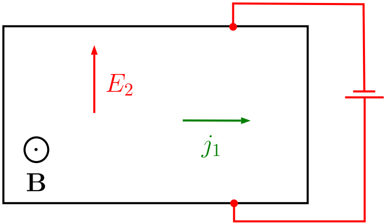

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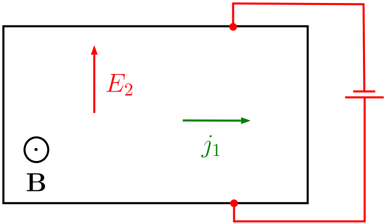

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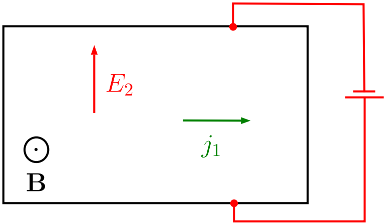

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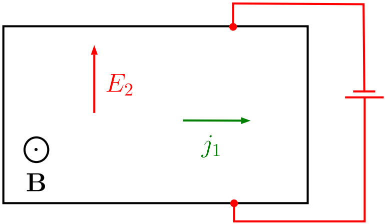

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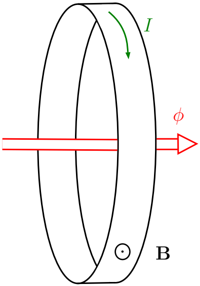

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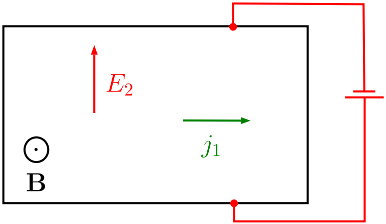

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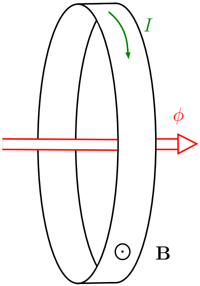

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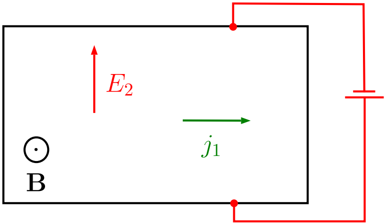

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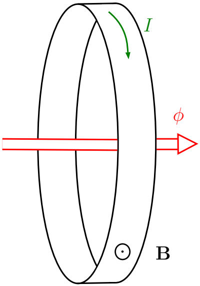

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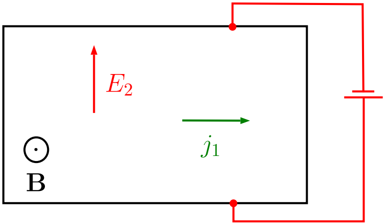

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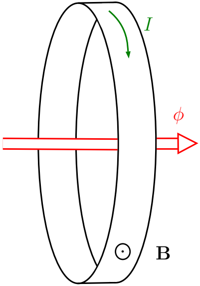

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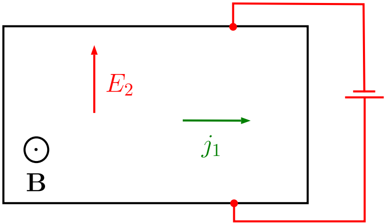

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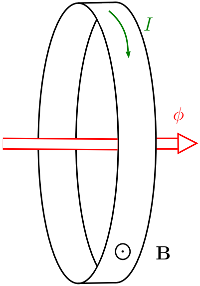

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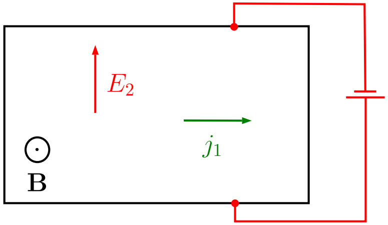

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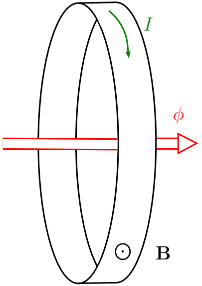

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Purely linear response of the quantum Hall current

In the interpretation of QHE as a bulk effect:

Theorem [G. M., D. Monaco] (informal statement)
For non-interacting, periodic electrons at zero temperature under a spectral gap assumption:

$$j_1 = \sigma_B E_2 + O(E_2{}^{\infty}).$$

Its proof is based on space-adiabatic perturbation theory [Kato, Nenciu, Teufel, . . .]

Comments on existing literature

In the interpretation of QHE as a pump effect:

- The analogous result (informal statement)

$$< I_2 > = \sigma_P V_1 + O(V_1{}^\infty)$$

  is due to [Klein, Seiler 1990] in the continuum (recently in the discrete [Bachmann, De Roeck, Fraas, Lange 2021]) setting for many-body electron gases at zero temperature under a spectral gas assumption.

- Their proofs is based on the physical *magnetic-flux insertion argument* by [Laughlin 1981], made rigorous by the use of the time-adiabatic perturbation theory [Avron, Seiler 1985; Avron, Seiler, Yaffe 1987; Avron, Seiler, Simon 1994, . . .].

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Review: Klein–Seiler's argument

On $\Lambda := [0, L_1] \times [0, L_2] \subset \mathbb{R}^2$ consider (a fermionic many-body version of)

$$\widetilde{H}(\phi_1, \phi_2) := \frac{1}{2}\left(\mathbf{p_A} - \phi_1 \frac{\mathbf{e}_1}{L_1} - \phi_2 \frac{\mathbf{e}_2}{L_2}\right)^2 + W(\mathbf{x})$$

with periodic boundary condition, where

- $\mathbf{p_A} := -\mathrm{i}\nabla - \mathbf{A}(\mathbf{x})$, where $\mathbf{A}$ models an external magnetic field
- $\phi_i \frac{\mathbf{e}_i}{L_i}$ is a vector potential "generating a magnetic flux ϕ_i through loop in the i-th direction"
- W stands for all interaction among the particles and of the particles with impurities
- Beyond regularity assumptions, suppose that $\widetilde{H}(\phi_1, \phi_2)$ has an isolated spectral component $\sigma_*(\phi_1, \phi_2)$, with corresponding finite rank spectral projection $\widetilde{P}(\phi_1, \phi_2)$

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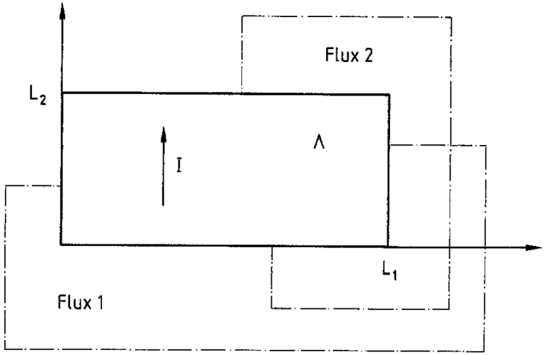
Geometry of the quantum Hall system:



Notice that $\widetilde{H}(\phi_1,\phi_2)$ is 2π-periodic in ϕ_1, ϕ_2 up to a gauge transformation: defining $G(\phi_1,\phi_2) := e^{i(\phi_1 X_1/L_1 + \phi_2 X_2/L_2)}$

$$\widehat{H}(\phi_1,\phi_2) := G^*(\phi_1,\phi_2)\widetilde{H}(\phi_1,\phi_2)G(\phi_1,\phi_2)$$
$$= \widehat{H}(\phi_1 + 2\pi,\phi_2) = \widehat{H}(\phi_1,\phi_2 + 2\pi)$$

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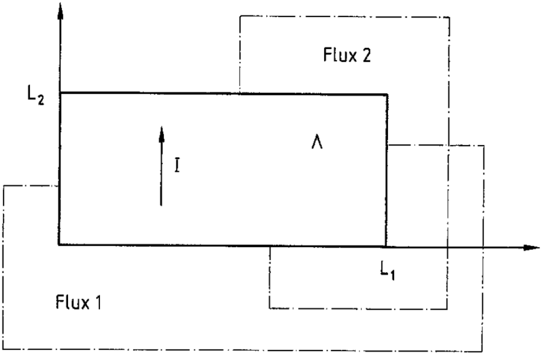

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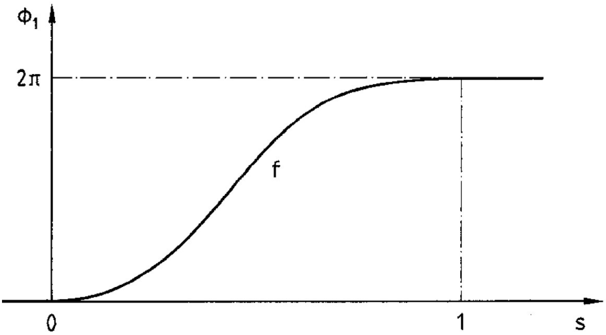
Time-dependent Hamiltonian

$$\widetilde{\widetilde{H}}_\tau(t,\Phi) := \widetilde{H}(\phi_1 = f(t/\tau), \phi_2 = \Phi), \qquad H_\tau(s,\Phi) := \widetilde{\widetilde{H}}_\tau(s\tau,\Phi)$$

- $\eta := \tau^{-1}$: time-adiabatic parameter ($\eta \ll 1$)
- $s := t/\tau$: scaled time
- $\phi_1'(t) = f'(s)\eta \propto V_1$: Hall voltage



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- Physical evolution:

$$\mathrm{i}\partial_s U_\tau(s,\Phi) = \tau\, H_\tau(s,\Phi)\, U_\tau(s,\Phi), \quad U_\tau(0,\Phi) = \mathrm{Id}$$

- Physical state:

$$P_\tau(s,\Phi) = U_\tau(s,\Phi) P(0,\Phi) U_\tau^*(s,\Phi), \quad P(s,\Phi) = \chi_{\sigma_*(s,\Phi)}(H(s,\Phi))$$

- Adiabatic evolution:

$$\mathrm{i}\partial_s U_{\mathrm{ad}}(s,\Phi) = \tau\, H_{\mathrm{ad}}(s,\Phi)\, U_{\mathrm{ad}}(s,\Phi), \quad U_{\mathrm{ad}}(0,\Phi) = \mathrm{Id}$$

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$$H_{\mathrm{ad}}(s,\Phi) := H_\tau(s,\Phi) + \frac{\mathrm{i}}{\tau}[\partial_s P(s,\Phi), P(s,\Phi)]$$

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Theorem (Adiabatic theorem)
Under previous assumptions, one has

- The adiabatic evolution intertwines the spectral subspaces:

$$U_{\mathrm{ad}}(s,\Phi)P(0,\Phi) = P(s,\Phi)U_{\mathrm{ad}}(s,\Phi)$$

- Since $H_\tau(s,\Phi)$ is constant near $s = 1$ $(\operatorname{supp} f' \subset (0,1))$

$$U^*_{\mathrm{ad}}(1,\Phi)U_\tau(1,\Phi)P(0,\Phi) = P(0,\Phi)U^*_{\mathrm{ad}}(1,\Phi)U_\tau(1,\Phi) + \mathcal{O}(\tau^{-\infty})$$

Review: Klein–Seiler's argument

The Hall current intensity

$$I_2(s,\Phi) := \mathrm{Tr}\left(P_\tau(s,\Phi)\partial_\Phi H_\tau(s,\Phi)\right).$$

The Φ-average transported charge

$$\begin{aligned}
\langle Q_2\rangle &:= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\Phi \left(\tau \int_0^1 \mathrm{d}s\, I_2(s,\Phi)\right) \\
&= \frac{\mathrm{i}}{2\pi}\int_0^{2\pi} \mathrm{d}\Phi \int_0^1 \mathrm{d}s\, \partial_s \mathrm{Tr}\left(P(0,\Phi) U_\tau^*(s,\Phi)\partial_\Phi U_\tau(s,\Phi)\right) \\
&= \frac{\mathrm{i}}{2\pi}\int_{\partial([0,1]\times[0,2\pi])} \mathrm{Tr}\left(P(0,\Phi) U_{\mathrm{ad}}^*(s,\Phi)\mathrm{d} U_{\mathrm{ad}}(s,\Phi)\right) + \mathscr{O}(\tau^{-\infty}) \\
&= \underbrace{\frac{1}{2\pi}}_{=e^2/h} \underbrace{\mathrm{i}\int_{\mathbb{T}^2} \mathrm{d}\phi_1 \mathrm{d}\phi_2\, \mathrm{Tr}\left(\widehat{P}[\partial_{\phi_1}\widehat{P}, \partial_{\phi_2}\widehat{P}]\right)}_{\text{Chern number } \in \mathbb{Z}} + \mathscr{O}(\tau^{-\infty})
\end{aligned}$$

where last equality uses the Chern–Simons formula:

$$\mathrm{Tr}\, P_U \mathrm{d}P_U \wedge \mathrm{d}P_U = \mathrm{Tr}\, P \mathrm{d}P \wedge \mathrm{d}P + \mathrm{d}\left(\mathrm{Tr}\, P U^{-1} \mathrm{d}U\right)$$

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Our argument

Assumption (H$_0$)

- $\mathscr{H} := L^2(\mathscr{X}) \otimes \mathbb{C}^N$,
  $\mathscr{X} = \mathbb{R}^d$ or $\mathscr{X} =$ discrete d-dimensional crystal $\subset \mathbb{R}^d$
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- H_0 is a periodic gapped operator on $\mathscr{H}$ and bounded from below
  - Bravais lattice of translations $\Gamma \simeq \mathbb{Z}^d$

    $$[H_0, T_\gamma] = 0 \quad \forall \gamma \in \Gamma$$

  - via Bloch–Floquet representation $H_0 \simeq \int_{\mathbb{T}^d}^{\oplus} \mathrm{d}k \, H_0(k)$,
    $H_0(k)$ acts on $\mathscr{H}_{\mathrm{f}} := L^2(\mathscr{C}_1) \otimes \mathbb{C}^N$, $\mathscr{C}_1 := \mathscr{X}/\Gamma$

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- H_0 is a periodic gapped operator on $\mathscr{H}$ and bounded from below, such that H_0 satisfies technical but mild hypotheses
  - $$H_0 : \mathbb{R}^d \to \mathscr{L}(\mathscr{D}_{\mathrm{f}}, \mathscr{H}_{\mathrm{f}}), \quad k \mapsto H_0(k)$$
    is a smooth equivariant map taking values in the self-adjoint operators with dense domain $\mathscr{D}_{\mathrm{f}} \subset \mathscr{H}_{\mathrm{f}}$. $\mathscr{L}(\mathscr{D}_{\mathrm{f}}, \mathscr{H}_{\mathrm{f}})$ is the space of bounded operators from $\mathscr{D}_{\mathrm{f}}$, equipped with the graph norm of $H_0(0)$, to $\mathscr{H}_{\mathrm{f}}$

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- in most tight-binding models having spectral gap (discrete case)

- by gapped, periodic Schrödinger operators

$$H_0 = \frac{1}{2}(-\mathrm{i}\nabla - A(x))^2 + V(x)$$

under standard hypotheses of relative boundedness of the potentials w.r.t. $-\Delta$ (continuum case)

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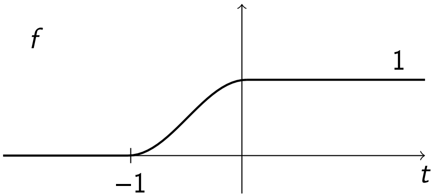
A model for the switching process

$$H_\varepsilon(t) := H_0 - \varepsilon f(t) X_2, \quad t \in I,$$

where $[-1,0] \subset I \subset \mathbb{R}$ is compact interval and $\varepsilon \ll 1$.



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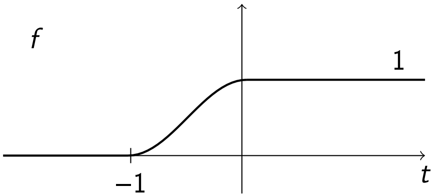
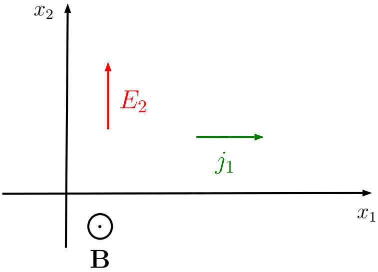


Our argument

Geometry of the quantum Hall system ($d = 2$ & continuum):



Notice that

- the configuration space is the plane $\mathbb{R}^2$ (while in the context of QHE as pump effect a cylindrical or torus geometry is necessary) and so the system has infinite extent (no thermodynamic limit needed)
- for every $\varepsilon > 0$ the domain $\mathscr{D}(H_\varepsilon) \neq \mathscr{D}(H_0)$ and the spectrum $\sigma(H_\varepsilon) = \mathbb{R}$

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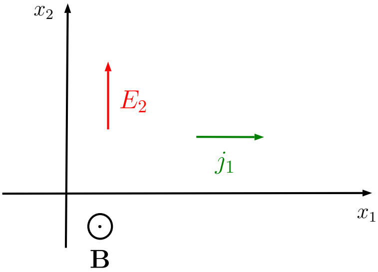

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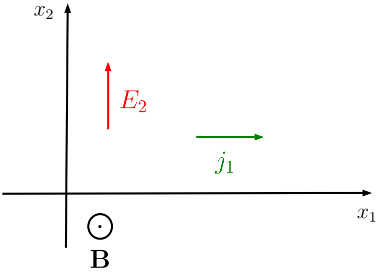

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A trace functional to compute expectation values of extensive observable in extended states (due to periodicity). Let $\Gamma = \mathbb{Z}^d$

- Trace per unit volume: For any A being trace class on compact sets,

$$\tau(A) := \lim_{\substack{L\to\infty \\ L\in 2\mathbb{N}+1}} \frac{1}{L^d} \mathrm{Tr}(\chi_L A \chi_L), \quad \chi_L := \chi_{[-L/2, L/2]^d}.$$

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- The physical state: $\rho(t)$ being the solution of the following Cauchy problem

$$\begin{cases} \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [H_\varepsilon(\eta t), \rho(t)] \\ \rho(t_0) = \Pi_0, \qquad \eta t_0 \leq -1. \end{cases}$$

  One is interested in $\rho(t) \equiv \rho_{\varepsilon,\eta}(t)$ for any $t \geq 0$ (when the perturbation is fully switched on).

- A *enough good* approximation of the physical state: non-equilibrium almost-stationary state (NEASS) $\Pi_{\varepsilon,n}$ such that for every $n, m \in \mathbb{N}$

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  Inequality $(\sharp)$ is proved for interacting models on lattices [Henheik, Teufel 2021, Teufel 2020, Monaco, Teufel 2019], while for this framework it is work in progress with Teufel.

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Construction of the NEASS at every order in ε

Consider the stationary perturbed Hamiltonian

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Theorem[G. M., D. Monaco]

Under Assumption (H_0), we have that for any $n \in \mathbb{N}$ there exists a unique NEASS $\Pi_{\varepsilon,n}$ such that

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where $\quad \mathscr{S}_{\varepsilon,n} := \sum_{j=1}^{n} \varepsilon^{j-1} A_j \quad$ and $\quad [H_\varepsilon, \Pi_{\varepsilon,n}] = \varepsilon^{n+1} [R_{\varepsilon,n}, \Pi_{\varepsilon,n}]$.

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The proof relies on the following

Lemma

Under Assumption (H_0), decompose $\mathscr{H} = \mathrm{Ran}\,\Pi_0 \oplus (\mathrm{Ran}\,\Pi_0)^\perp$ and correspondingly operators as

$$A = A^{\mathrm{D}} + A^{\mathrm{OD}}$$

where $A^{\mathrm{D}} = \Pi_0 A \Pi_0 + \Pi_0^\perp A \Pi_0^\perp$, $A^{\mathrm{OD}} = \Pi_0 A \Pi_0^\perp + \Pi_0^\perp A \Pi_0$.
Define the *Liouvillian*

$$\mathscr{L}_{H_0}(A) = -\mathrm{i}[H_0, A].$$

Then the Liouvillian is invertible on OD operators (thanks to the gap condition):
For any $B = B^{\mathrm{OD}}$, consider $\mathscr{L}_{H_0}(A) = B$

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For example

$$\mathscr{S}_{\varepsilon,1} = -\mathscr{L}_{H_0}^{-1}([[X_2, \Pi_0], \Pi_0]) = -\mathscr{L}_{H_0}^{-1}(X_2^{\mathrm{OD}}).$$

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Purely linear response of the quantum Hall current to space-adiabatic perturbations

Theorem[G. M., D. Monaco]
Consider the Hamiltonian $H_\varepsilon = H_0 - \varepsilon X_2$, where H_0 satisfies Assumption (H_0). Then for every $n \in \mathbb{N}$ we have that

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where $\Pi_{\varepsilon,n}$ is as in the previous Theorem and

$$\sigma_{\mathrm{Hall}} := \mathrm{i}\tau(\Pi_0 [[\Pi_0, X_1], [\Pi_0, X_2]]).$$

Remark
Up to prove the validity of the NEASS approximation for the state of the system (in the sense of inequality (♯)),

$$\tau(J_1 \rho_{\varepsilon,\eta}(t)) = \varepsilon\, \sigma_{\mathrm{Hall}} + \mathscr{O}(\varepsilon^{n+1}), \qquad t \geq 0$$

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Sketch of the proof

Let's recall $J_1 \, \Pi_{\varepsilon,n} = \mathrm{i}[H_0, X_1] \, \Pi_{\varepsilon,n}$

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Sketch of the proof

By using the cyclicity of $\tau(\cdot)$ and $(\Pi_{\varepsilon,n})^2 = \Pi_{\varepsilon,n}$

$$\tau\left([H_0, X_1]\Pi_{\varepsilon,n}\right) = \tau\left(\Pi_{\varepsilon,n}[H_\varepsilon, X_1]\Pi_{\varepsilon,n}\right)$$

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In view of $[H_\varepsilon, \Pi_{\varepsilon,n}] = \varepsilon^{n+1}[R_{\varepsilon,n}, \Pi_{\varepsilon,n}]$

$$\tau\left([H_0, X_1]\Pi_{\varepsilon,n}\right) = \tau\left(\Pi_{\varepsilon,n}[H_\varepsilon, X_1]\Pi_{\varepsilon,n}\right)$$
$$= \tau\left([\Pi_{\varepsilon,n} H_\varepsilon \Pi_{\varepsilon,n}, \Pi_{\varepsilon,n} X_1 \Pi_{\varepsilon,n}]\right) + \varepsilon^{n+1}\, \tau\left(\Pi_{\varepsilon,n}\big[[\Pi_{\varepsilon,n}, R_{\varepsilon,n}], [X_1, \Pi_{\varepsilon,n}]\big]\Pi_{\varepsilon,n}\right)$$

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By using $H_\varepsilon := H_0 - \varepsilon X_2$

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We conclude noticing that $\tau\left([\Pi_{\varepsilon,n}H_0\Pi_{\varepsilon,n}, \Pi_{\varepsilon,n}X_1\Pi_{\varepsilon,n}]\right) = 0$ by cyclicity of the trace, and the *Chern–Simons-like formula* defining $P_U := UPU^{-1}$, one has that $\tau([P_U X_1 P_U, P_U X_2 P_U]) = \tau([PX_1P, PX_2P])$ for U, P periodic and *regular enough*.

What next?

- Validity of the NEASS approximation for the physical state in one-body approximation in the continuum (sub-project: energy and space estimates for the physical evolution, similar to [M. 2022])

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