General bulk-edge correspondence at positive temperature

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TÜBINGEN

What is the bulk-edge correspondence?

Physically:
Integer Quantum Hall Effect.

Linear response theory in the infinite volume limit gives $\sigma_H \in \mathbb{Z}$.

Analysis of “edge modes/currents” at the boundary of the sample gives $\sigma_E \in \mathbb{Z}$

At zero temperature, with (mobility) gap

$$\sigma_H = \sigma_E \in \mathbb{Z}$$

Mathematically:

Bulk system defined on $L^2(\mathbb{R}^2)$.

Edge system defined by cutting the bulk one and imposing Dirichlet boundary conditions.

Is there any mathematical relation/correspondence between the two systems?

Our goal: Prove bulk-edge correspondence (for transport coefficients) at any temperature.

$\Rightarrow$ Longer route than expected!

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The bulk dynamics is described by a magnetic random Schroedinger operator on $L^2(\mathbb{R}^2)$:

$$H_{\omega,b} = \frac{1}{2}\left(-\mathrm{i}\nabla - \mathcal{A} - bA\right)^2 + V + V_\omega$$

Let $\tau_{b,\gamma}$ be a family of magnetic translations compatible with the Landau gauge, and $T(\gamma)$ the canonical action of $\mathbb{Z}^2$ on Θ ($T(\gamma)\omega = \{\omega_{\eta-\gamma}\}_{\eta\in\mathbb{Z}^2}$)

$$\Rightarrow \quad \tau_{b,\gamma} H_{\omega,b} \tau_{b,-\gamma} = H_{T(\gamma)\omega,b}, \quad \forall \gamma \in \mathbb{Z}^2 .$$

$\Rightarrow$ $(H_{\omega,b})_{\omega\in\Theta}$ is ergodic with respect to the lattice $\mathbb{Z}^2$

$\rightarrow$ No assumption on the spectrum of the model!

Mathematical framework - The bulk-edge model

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- Scalar potential V and magnetic potential $\mathcal{A}$ are smooth and $\mathbb{Z}^2$ periodic, namely

$$V(\mathbf{x}+\gamma) = V(\mathbf{x})\,, \quad \mathcal{A}(\mathbf{x}+\gamma) = \mathcal{A}(\mathbf{x}) \qquad \gamma \in \mathbb{Z}^2\,.$$

- $\mathbb{R} \ni b := -e\mathfrak{B}$ and $\mathbf{A}$ is the magnetic potential in the Landau gauge

$$A = (-x_2, 0)\,.$$

- The disordered background is modelled by the usual Anderson potential given by independent identically distributed random variables:

$$\{\omega_\gamma\}_{\gamma\in\mathbb{Z}^2} = \omega \,\in\, \Theta = [-1,1]^{\mathbb{Z}^2}\,, \quad \mathbb{P} = \bigotimes_{\mathbb{Z}^2} \mu$$

$$V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^2} \omega_\gamma u(x-\gamma) \quad u \in C_0^\infty(\mathbb{R}^2)$$

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Consider the half-plane

$$E := \{(x_1, x_2) \in \mathbb{R}^2 | \, x_2 \geq 0\}.$$

The edge dynamics is described by the Hamiltonian $H^E_{\omega,b}$ living in $L^2(E)$ $\rightarrow$ $H^E_{\omega,b}$ is the natural choice given by the Dirichlet realization of $H_{\omega,b}$ in E
$\rightarrow$ we cut the bulk system.

$(H^E_{\omega,b})_{\omega\in\Theta}$ is still ergodic with respect to the one-dimensional lattice generated by the vector $(1,0)$:

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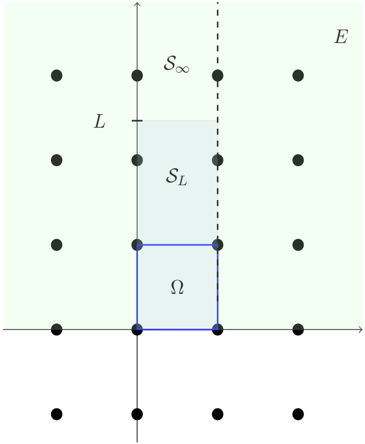

$$\Omega := [0,1]^2$$

is the unit cell of the bulk Hamiltonian and χ_Ω is characteristic function of Ω.

$$\mathcal{S}_L := [0,1] \times [0,L]$$

χ_L characteristic function of $\mathcal{S}_L$.

$$\mathcal{S}_\infty := [0,1] \times [0,\infty]$$

χ_∞ characteristic function of $\mathcal{S}_\infty$.

(Edge) Thermodynamic pressure

Let $F_{\mu,T}(x) = -T \ln \left(1 + \mathrm{e}^{-(x-\mu)/T}\right)$ be the grandcanonical potential.
Remember that $F'_{\mu,T}(x) = \frac{1}{\mathrm{e}^{(x-\mu)/T}+1}$ is the Fermi-Dirac distribution

Bulk pressure

The bulk pressure is defined as the thermodynamic limit of the density of grandcanonical potential

$$p_{\mu,T}(b) := -\mathbb{E}\left(\mathrm{Tr}\left(\chi_\Omega F_{\mu,T}(H_{\bullet,b})\right)\right) = -\lim_{L\to\infty} \frac{1}{L^2} \mathrm{Tr}(\chi_{\Lambda_L} F_{\mu,T}(H_{\omega,b})) \quad \text{for a.e. } \omega.$$

What about the edge?

Edge pressure

$$p^{(E)}_{\mu,T}(b) := -\lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) := -\lim_{L\to\infty} \frac{1}{L} \mathrm{Tr}\left(\chi_L F_{\mu,T}(H^E_{\omega,b})\right) \quad \text{for a.e. } \omega.$$

$\to$ These are the only two ingredients that we need!

Bulk-edge correspondence at positive temperature

Theorem [H. Cornean, M.M., S.Teufel]

First, $p_{\mu,T}(\cdot)$ and $P_{\mu,T}^{(L,\omega)}(\cdot)$ are everywhere differentiable and, for a.e. $\omega \in \Theta$:

$$p_{\mu,T}^{(E)}(b) = \lim_{L\to\infty} P_{\mu,T}^{(L,\omega)}(b) = p_{\mu,T}(b), \qquad \lim_{L\to\infty} \frac{\mathrm{d}P_{\mu,T}^{(L,\omega)}}{\mathrm{d}b}(b) = \frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b).$$

Moreover, let $g \in C^1([0,1])$ be any function such that $g(0) = 1$ and $g(1) = 0$. Define $\widetilde{\chi}_L(\mathbf{x}) := \chi_L(\mathbf{x})g(x_2/L)$. Then independently of g we have:

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} \mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H_{\bullet,b}^E, X_1\right] F'_{\mu,T}(H_{\bullet,b}^E)\right\}\right). \qquad (\star)$$

- $(\star)$ holds true at every temperature.
- $(\star)$ holds independently of the spectrum of $H_{\omega,b}$.
- Purely analytic proof.
- Stability w.r.t. boundary perturbations.

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The main result

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still hold true in the case where the edge Hamiltonian is perturbed by a smooth potential W_ω supported in a finite strip near the edge.

Scalar potential W_ω, s.t. $\mathrm{supp}(W_\omega) \subseteq \mathbb{R} \times [0, d]$, $d > 0$.
$H^{E,W}_{\omega,b} = H^E_{\omega,b} + W_\omega$, densely defined on $L^2(E)$ with Dirichlet boundary condition at $x_2 = 0$.
Assume that $(H^{E,W}_{\omega,b})_{\omega\in\Theta}$ is still ergodic on the one-dimensional lattice generated by $(1, 0)$.

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Physical interpretation: the bulk side

Left-hand side (the bulk):

$$-e\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = m_{\mu,T}(b)$$

is just the definition of the bulk magnetization.

What about the right-hand side (the edge)?

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$$j_1^B(x_2) := \int_0^1 \mathrm{d}x_1\, \mathbb{E}\left(\mathrm{i}\,[H_{b,\bullet}, X_1]\, F'_{\mu,T}(H_{b,\bullet})\right)(x_1, x_2; x_1, x_2)$$

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Theorem [M.M., B. Støttrup]

j_1^E and j_1^B are smooth functions in $\mathbb{R} \times (0, +\infty)$ and

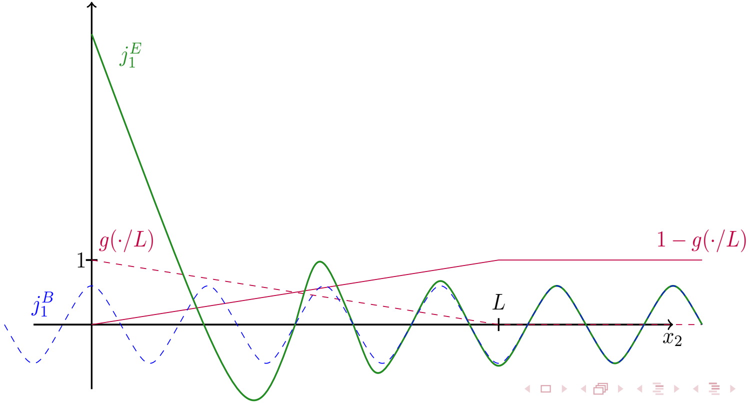
$$j_1^E(x_2) - j_1^B(x_2) = \mathcal{O}(x_2^{-\infty}) \qquad x_2 \to +\infty$$

The edge side

The total edge current is defined as

$$I_1^E(\mu,T,b) := \lim_{L\to\infty}\int_0^L \big(j_1^E(x_2) - \big(1-g(x_2/L)\big)j_1^B(x_2)\big)\,\mathrm{d}x_2$$

$$= \lim_{L\to\infty}\int_0^L g(x_2/L)j_1^E(x_2)\,\mathrm{d}x_2\,.$$



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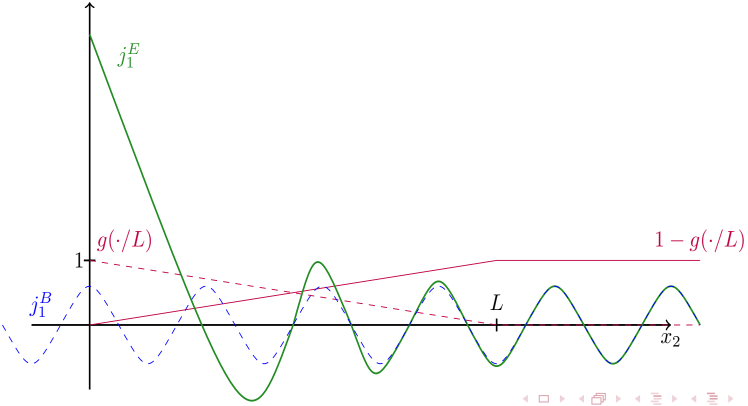
$\rightarrow$ We show that the value of I_1^E is actually **independent of the specific cut-off function** g and **of the specific potential at the boundary** ! $\rightarrow$ It is a very robust quantity that lives near the edge!



Physical interpretation II

Therefore we get the bulk-edge correspondence in the form:

$$m_{\mu,T}(b) = -eI_1^E(\mu, T, b).$$

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Zero-temperature limit and bulk-edge correspondence

At positive temperature the pressure is C^2 in b and μ (Briet-Savoie RMP12):

$$\partial_\mu p_{\mu,T}(b) = n_{\mu,T}(b) = \mathbb{E}(\mathrm{Tr}(\chi_\Omega F'_{\mu,T}(H_{\omega,b})))$$

where $n_{\mu,T}(b)$ is the particle density.

$$\Rightarrow \partial_\mu m_{\mu,T}(b) = \partial_\mu \partial_b p_{\mu,T}(b) = \partial_b n_{\mu,T}(b)$$

Assume that the **almost sure spectrum** $\Sigma(b_0)$ of the bulk Hamiltonian H_{ω,b_0}, $b_0 \in \mathbb{R}$, **has a gap** that includes the interval $[e_-, e_+](\ni \mu)$ with $e_- < e_+$.
$\rightarrow \sigma_0(b) := \Sigma(b) \cap (-\infty, e_-)$, $P_{\omega,b}$ the spectral projection onto $\sigma_0(b)$.

Středa formula [Cornean,Monaco, M.M. JEMS 21, ...]

$$2\pi\partial_b n_{\mu,0}(b_0) = C_0 := 2\pi\mathbb{E}\Big(\,\mathrm{Tr}\left(\chi_\Omega P_{\bullet,b_0}\mathrm{i}\big[[X_1, P_{\bullet,b_0}],[X_2,P_{\bullet,b_0}]\big]\right)\Big) = \sigma_H \; (\in \mathbb{Z})$$

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Key ingredients: **Středa formula** + **Zero-temperature Hall conductivity**

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Proposition [H. Cornean, M.M., S. Teufel]

There exist two constants $C_1, C_2 > 0$ such that

$$\left|\partial_b n_{\mu,T}(b_0) - \sigma_H(\mu,0,b)\right| \le C_1 \, \mathrm{e}^{-C_2/T}$$
$$\left|\partial_\mu I^E(\mu,T,b_0) - \sigma_E(\mu,0,b)\right| \le C_1 \, \mathrm{e}^{-C_2/T}.$$

Moreover, let χ_∞ denote the indicator function of the strip $\mathcal{S}_\infty := [0,1] \times (0,\infty)$, then, independently of the specific choice f_0, we have:

$$\lim_{T \searrow 0} \partial_b n_{\mu,T}(b_0) = \sigma_H(\mu,0,b) = \mathbb{E}\left(\mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E_{\bullet,b_0}, X_1\right] f_0'(H^E_{\bullet,b_0})\right\}\right) = \sigma_E(\mu,0,b).$$

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New approach to bulk-edge correspondence based on magnetic perturbation theory
$\rightarrow$ Magnetic derivative of bulk/edge quantity.
Versatile approach suitable for the many-body setting
(Work in progress with J. Lampart, S. Teufel, T. Wessel)

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$\rightarrow$ Problems:

- Středa formula holds true only at zero temperature and with μ in a spectral gap.
- Linear response theory at positive temperature? (Aizenman-Graf 1996; Cornean-Nenciu-Pedersen 2010)

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Heuristic physical picture

First emphasized by Haidu-Gummich ’83, Cooper-Halperin-Ruzin ’96, Středa 2006 -> Elgart-Graf-Schenker 2005

“*For a quantum mechanical system in the presence of an applied magnetic field, however, there may be nonzero circulating currents even in a situation of thermodynamic equilibrium, as was noted above. We shall find it convenient to break the currents into a “transport” part and a “magnetization” part...*”

Cooper, Halperin, Ruzin. Phys. Rev. B 1996

Splitting the edge current density

$$j^E(x) = j^E_{mag}(x) + j^E_{tr}(x)$$

Splitting the magnetization

$$m_{\mu,T}(b) = m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

j^E_{mag} is a pure "magnetization current density", that is

$$j^E_{mag}(x) := \nabla \times m^{(circ)}_{\mu,T}(x).$$

→ Why this magnetization current influences only the edge?

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$$m_{\mu,T}(b) \approx \frac{1}{|\Lambda_L|} \mathrm{Tr}\Big((\partial_b H_\Lambda(b)) F'_{\mu,T}(H_\Lambda(b)) \Big)$$
$$= -\frac{1}{|\Lambda_L|} \int_\Lambda dx\, x_2 \Big(P_1(b)\; F'_{\mu,T}(H_\Lambda(b)) \Big)(x,x).$$

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j^E_{mag} is a pure "magnetization current density", that is

$$j^E_{mag}(x) := \nabla \times m^{(circ)}_{\mu,T}(x)$$

Heuristic physical picture

Splitting the edge current density

$$j^E(x) = j^E_{mag}(x) + j^E_{tr}(x)$$

$$m_{\mu,T}(b) \approx \frac{1}{|\Lambda_L|} \mathrm{Tr}\Big((\partial_b H_\Lambda(b)) F'_{\mu,T}(H_\Lambda(b)) \Big)$$
$$= -\frac{1}{|\Lambda_L|} \int_\Lambda dx \; x_2 \Big(P_1(b) \; F'_{\mu,T}(H_\Lambda(b)) \Big)(x,x).$$

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$$\Rightarrow I^E = I^E_{mag} + I^E_{tr} = m^{(circ)}_{\mu,T}(b) + m^{(res)}_{\mu,T}(b) = m_{\mu,T}(b)$$

In order to get the correct transport edge current we have to be able to split either the current or the magnetization!

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Bulk-edge correspondence $T \geq 0$: Landau case

Setting:
Bulk Hamiltonian $H_b = \frac{1}{2}(-\mathrm{i}\nabla - bA)^2$
Spectrum given by infinitely degenerate eigenvalue $\left\{E_{n,b} = b(n + \frac{1}{2}) \,|\, n \in \mathbb{N}\right\}$.
$\Pi_{n,b}$ spectral projection onto $E_{n,b}$.
Integrated density of states associated to each Landau level:

$$\lim_{L \to \infty} \frac{\mathrm{Tr}(\chi_{\Lambda_L} \Pi_{n,b})}{L^2} = \mathrm{Tr}(\chi_\Omega \Pi_{n,b}) = \frac{b}{2\pi}.$$

Hall conductivity for $T \geq 0$ (Evaluation of Kubo formula, Cornean-Nenciu-Pedersen 2006, physics paper...):

$$\sigma_H(\mu, T, b) = -\frac{n_{\mu,T}(b)}{b}.$$

$\to$ The pressure is simply given by

$$p_{\mu,T}(b) = -\sum_{n=0}^{\infty} F_{\mu,T}(E_{n,b})\,\mathrm{Tr}(\chi_\Omega \Pi_{n,b}) = -\sum_{n=0}^{\infty} F_{\mu,T}(E_{n,b}) \frac{b}{2\pi}$$

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$$=: m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

$$\Rightarrow (\partial_b n_{\mu,T}(b) =)\partial_\mu m_{\mu,T}(b) = \boxed{\sum_{n=1}^{\infty} F''_{\mu,T}(E_{n,b})\frac{dE_{n,b}}{db}\frac{b}{2\pi}} + \sigma_H(\mu, T, b)$$

$\rightarrow$ Extra term $\rightarrow$ the weighted sum of the angular momentum of each states in the Landau levels!

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$$m^{circ}_{\mu,T}(b) := \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{d}{db} E_{n,b} \frac{b}{2\pi}$$

is magnetic moment per unit area.

⇒ $m^{circ}_{\mu,T}(b)$ is the part of the magnetization given by the local circulation
⇒ we have to subtract the associated edge current contribution:

$$m^{circ}_{\mu,T}(b) = I_{mag}(\mu,T,b)$$

Our formula $(\star)$

$$\Rightarrow \boxed{m_{\mu,T}(b) = I^E(\mu,T,b)}$$

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$$\sigma_H(\mu,T,b) = -\frac{n_{\mu,T,b}}{b} = \partial_\mu m^{res}_{\mu,T}(b) = \partial_\mu I^E_{tr}(\mu,T,b) = \sigma_E(\mu,T,b)$$

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- $\frac{b}{2\pi}$ is the number of states per unit area.
- $F'_{\mu,T}(E_{n,b})$ is the statistical weight.
- What about $\frac{d}{db} E_{n,b} = n$?
  $\rightarrow$ "Hellmann-Feynman theorem", $\{\psi_{n,m}\}_{m \in \mathbb{Z}}$ o.n.b. for $\mathrm{Ran}\Pi_{n,b}$

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Isolated simple Bloch bands

What about the splitting of the magnetization in (more) general situations?

Theorem [Schulz–Baldes-Teufel 12 ; Teufel-Stiepan 13]

In tight-binding model, with M simple isolated Bloch bands we have:

$$m_{\mu,T}(b) = \sum_{l=1}^{M} \int_{\mathbb{B}_b} \frac{dk}{(2\pi)^d} \left(F'_{\mu,T}(E_l(k))\, R^{(l)}_{j+1,j+2}(k) + F_{\mu,T}(E_l(k))\Omega^{(l)}_{1,2}(k) \right)$$

$$=: m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

Conjecture [See Bellissard-vanElst-Schulz–Baldes;Niu et al.;Resta et al.,]

$$\sigma_H(\mu,T,b) = \partial_\mu m^{res}_{\mu,T}(b)$$

$\Rightarrow$ the conjecture coupled with our formula $(\star)$ would imply bulk-edge correspondence at $T \geq 0$.

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where $R^{(l)}_{1,2}$ is the Rammal-Wilkinson tensor and $\Omega^{(l)}_{1,2}(k)$ is the Berry curvature:

$$R^{(l)}_{1,2}(k) = \frac{\imath}{2} \left(\mathrm{Tr}\left(P_l(k) \partial_1 P_l(k) \left(H(k) - E_l(k)\right) \partial_2 P_l(k)\right) - (1 \leftrightarrow 2)\right)$$

$$\Omega^{(l)}_{1,2}(k) = \imath \, \mathrm{Tr}\left(P_l(k) \left[\partial_1 P_l(k), \partial_2 P_l(k) \right]\right)$$

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Sketch of the proof

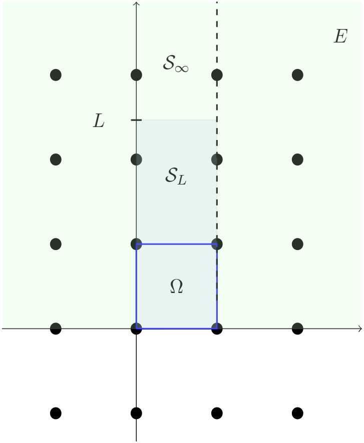
Theorem [H. Cornean, M.M., S.Teufel]

First, $p_{\mu,T}(\cdot)$ and $P^{(L,\omega)}_{\mu,T}(\cdot)$ are everywhere differentiable and, for a.e. $\omega \in \Theta$:

$$p^{(E)}_{\mu,T}(b) = \lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = p_{\mu,T}(b), \qquad \lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}}{\mathrm{d}b}(b) = \frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b).$$

Moreover, let $g \in C^1([0,1])$ be any function such that $g(0) = 1$ and $g(1) = 0$. Define $\widetilde{\chi}_L(\mathbf{x}) := \chi_L(\mathbf{x})g(x_2/L)$. Then independently of g we have:

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} \mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$



$$\Omega := [0,1]^2$$

is the unit cell of the bulk Hamiltonian and χ_Ω is characteristic function of Ω.

$$\mathcal{S}_L := [0,1] \times [0,L]$$

χ_L characteristic function of $\mathcal{S}_L$.

$$\mathcal{S}_\infty := [0,1] \times [0,\infty]$$

χ_∞ characteristic function of $\mathcal{S}_\infty$.

Sketch of the proof

Step 0.: Trace class properties, regularities of integral kernels and vanishing of equilibrium "current", that is

$$\mathbb{E}\left(\mathrm{Tr}\left(\chi_\Omega \mathrm{i}[H_{\bullet,b}, X_i] F'(H_{\bullet,b})\right)\right) = 0, \qquad i \in \{1,2\}.$$

Step 1. Edge pressure coincides in the limit with the bulk pressure :

$$\lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = \lim_{L\to\infty} \frac{1}{L}\,\mathrm{Tr}\left(\chi_L F(H^E_{\omega,b})\right) = p_{\mu,T}(b)\,.$$

→ Main tool: geometric perturbation theory:

$L \geq 1$; Strip near the edge: $\Xi_L(t) := \left\{\mathrm{x} \in E \,|\, \mathrm{dist}(\mathrm{x}, \partial E) \leq t\sqrt{L}\right\}$, $t > 0$.
$0 \leq \eta_0, \eta_L \leq 1$, smooth functions and only depending on x_2 such that $\eta_0(\mathrm{x}) + \eta_L(\mathrm{x}) = 1$ for every $\mathrm{x} \in E$

Take $0 \leq \widetilde{\eta}_0, \widetilde{\eta}_L \leq 1$ again only depending on x_2, such that they are a sort of stretched version of η_0 and η_L (no partition of unity in this case).

$$U_{L,\omega}(z) := \widetilde{\eta}_L \left(H_{\omega,b} - z\right)^{-1} \eta_L + \widetilde{\eta}_0 \left(H^E_{\omega,b} - z\right)^{-1} \eta_0.$$

$$W_{L,\omega}(z) := \left(-2\mathrm{i}\nabla\widetilde{\eta}_L \cdot (-\mathrm{i}\nabla - \mathcal{A} - bA) - (\Delta\widetilde{\eta}_L)\right)\left(H_{\omega,b} - z\right)^{-1} \eta_L$$

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$\rightarrow$ Main difficulty/novelty: F does not have compact support (F is a Scwhartz function and the spectrum is only unbounded from below)!

$\rightarrow$ Main tool: Helffer-Sjöstrand formula

$$F(H_\omega^{E/\ }) = -\frac{1}{\pi} \int_{\mathbb{R}\times[-1,1]} \mathrm{d}z_1 \mathrm{d}z_2 \, \bar{\partial} F_N(z) (H_\omega^{E/\ } - z)^{-1}, \quad z = z_1 + \mathrm{i}z_2,$$

F_N is an almost analytic extension of F, that is: Let $0 \leq g(y) \leq 1$ with $g \in C_0^\infty(\mathbb{R})$ such that $g(y) = 1$ if $|y| \leq 1/2$ and $g(y) = 0$ if $|y| > 1$. Fix some $N \geq 2$ and define

$$F_N(z_1 + \mathrm{i}z_2) := g(z_2) \sum_{j=0}^{N} \frac{1}{j!} \frac{\partial^j F}{\partial {z_1}^j}(z_1)(\mathrm{i}z_2)^j.$$

+ regularity and decay estimates on $\left(H_{\omega,b}^{\ /E} - z\right)^{-1}$.

Sketch of the proof

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$L \geq 1$; Strip near the edge: $\Xi_L(t) := \left\{ \mathbf{x} \in E \,|\, \mathrm{dist}(\mathbf{x}, \partial E) \leq t\sqrt{L} \right\}$, $t > 0$.

$0 \leq \eta_0, \eta_L \leq 1$, smooth functions and only depending on x_2 such that $\eta_0(\mathbf{x}) + \eta_L(\mathbf{x}) = 1$ for every $\mathbf{x} \in E$

Take $0 \leq \widetilde{\eta}_0, \widetilde{\eta}_L \leq 1$ again only depending on x_2, such that they are a sort of stretched version of η_0 and η_L (no partition of unity in this case).

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The resolvent of the edge Hamiltonian obeys the identity:

Sketch of the proof

Step 0.: Trace class properties, regularities of integral kernels and vanishing of equilibrium "current"

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$$\begin{aligned} &\operatorname{supp}(\eta_0) \subset \Xi_L(2), \\ &\operatorname{supp}(\eta_L) \subset E \setminus \Xi_L(1), \\ &\|\partial_2^n \eta_i\|_\infty \simeq L^{-\frac{n}{2}}, \quad n \geq 1, \quad i \in \{0, L\}\,. \end{aligned}$$

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Step 2. The magnetic derivative of the edge pressure has a thermodynamic limit:

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→ Magnetic derivative and thermodynamic limit commute!

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Modified asymmetric phase $\varphi(\mathbf{x},\mathbf{y}) := (y_1 - x_1)y_2$

$$(P_{x_1} - \epsilon A_1(\mathbf{x}))\mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})} = \mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})}(P_{x_1} - \epsilon A_1(\mathbf{x}-\mathbf{y})).$$

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Recap:

- Bulk-edge correspondence in the form $m = I^E$ at every temperature.
- Usual bulk-edge correspondence for μ in a gap and limit $\searrow T = 0$.
- Bulk-edge correspondence for the Landau Hamiltonian at $T \geq 0$.

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
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Thank you for your attention!

Cornean H.D., M. M., Teufel, S.: General bulk-edge correspondence at positive temperature. ArXiv: 2107.13456 (2021).

M.M, B. B. Støttrup: Regularity properties of bulk and edge current densities at positive temperature . ArXiv: 2201.08803 (2022).

Quantization of edge conductance at zero temperature

Let $V_\omega = 0 \to$ The Hamiltonian H_{b_0} commutes with magnetic translation (rationality condition on the magnetic field).

$\to$ the operator $\mathrm{i}\left[H^E_{b_0}, X_1\right] F'(H^E_{b_0})$ is $\mathbb{Z}$-periodic in the x_1-direction up to a Bloch-Floquet-Zak transform it is unitarily equivalent to $\int^{\oplus}_{[-\pi,\pi]} \mathrm{d}k_1\, h^E(k_1)$, where the fiber operator

$$h^E(k_1) = \frac{1}{2}(-\mathrm{i}\partial_{x_1} - \mathcal{A}_1(x_1,x_2) + b_0 x_2 + k_1)^2 + \frac{1}{2}(-\mathrm{i}\partial_{x_2} - \mathcal{A}_2(x_1,x_2))^2 + V(x_1,x_2)$$

is densely defined in $L^2(\mathcal{S}_\infty)$ with periodic boundary conditions on the lateral lines $x_1 \in \{0,1\}$ restricted to E, and with a Dirichlet boundary condition at the bottom $x_2 = 0$.

The bulk operator H_{b_0} can be written in similar manner, but its fiber $h(k_1)$ will be defined in $L^2([0,1] \times \mathbb{R})$ with periodic boundary conditions on the infinite lines $x_1 \in \{0,1\}$.

Quantization of edge conductance at zero temperature

- $h(k_1)$ does not have spectrum inside the gap $[e_-, e_+]$.
- $(h^E(k_1) - z)^{-1} - \chi_\infty (h(k_1) - z)^{-1} \chi_\infty$ is compact for any z with $\mathrm{Im}(z) \neq 0$, thus the spectral projection of $h^E(k_1)$ inside $[e_-, e_+]$ is compact, hence $h^E(k_1)$ can only have finitely many discrete eigenvalues which can be inside $[e_-, e_+]$.
- These eigenvalues are the so-called *edge states*, corresponding to eigenfunctions exponentially localized near the boundary $x_2 = 0$. According to Rellich's theorem, each edge state eigenvalue $\lambda(k_1)$ can be analytically followed as a function of k_1 as long as its value belongs to $[e_-, e_+]$.

Quantization of edge conductance at zero temperature

Lemma

Let $N < \infty$ be the total number of edge state eigenvalues which can enter the interval $[e_-, e_+]$. Without loss of generality we may assume that no such eigenvalue starts or ends at $e_\pm$, i.e. $\lambda_n(\pm\pi) \notin \{e_-, e_+\}$. Then:

$$
\begin{aligned}
& -2\pi \mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E, X_1\right] F_0'(H^E)\right\} = \sum_{n=1}^{N} \int_{-\pi}^{\pi} \mathrm{d}k_1 \; F_0'(\lambda_n(k_1))\lambda_n'(k_1) \\
& = \sum_{n=1}^{N} \{F_0(\lambda_n(-\pi)) - F_0(\lambda_n(\pi))\},
\end{aligned}
$$

and the right-hand side is an integer.