Evolution of Laughlin states under geometry deformation

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Plan of the talk

- Geometric quantization and the quantum Hall effect
- Evolution of quantum states in geometric quantization
- Application to the QHE on the sphere
- Final conclusions

Geometric quantization and the quantum Hall effect

As is well known, 1-particle states for the QHE on a 2-dimensional surface S correspond to the quantum states of the holomorphic quantization of the surface with symplectic form given by the magnetic field

$$\omega = B.$$

Thus, the physical configuration space S becomes an effective phase space (S, ω).

We will consider the case where ω is also the Kähler form, so that the Riemannian metric on S is given by

$$\gamma(\cdot, \cdot) = \omega(\cdot, J\cdot)$$

where J is the complex structure.

In this talk, I will describe how Hamiltonian flows in imaginary time it generate natural families of Kähler structures $(S, \omega, J_t, \gamma_t)$, and how 1-particle states evolve along this family of geometries of S. This will be applied to Laughlin states for the FQHE on the sphere S^2.

Evolution of quantum states in geometric quantization

Let (M, ω) be a symplectic manifold, $\dim M = 2n$.

Morally, the process of "quantization" should assign to it a Hilbert space $\mathcal{H}$, such that functions $f \in C^\infty(M)$ are promoted to operators $\hat{f}$ acting on $\mathcal{H}$ with

$$\widehat{\{f, g\}}_{\text{P.B.}} = \frac{i}{\hbar}[\hat{f}, \hat{g}], \quad f, g \in C^\infty(M),$$

along with a few other natural conditions including the irreducibility of this representation $\mathcal{H}$. It is known that this problem has no solution if one imposes all these requirements.

Geometric quantization is a rich framework where one can study mathematical issues related to the problem of quantization.

Assume that there exists $L \to M$ with Hermitian structure h and compatible connection ∇, with curvature $F_\nabla = -i\omega$. One calls (M, L, h, ∇) the *prequantization data.*

The prequantum Hilbert space $\Gamma_{L^2}(M, L)$ is too big for irreduciblity. Choose a polarization

$$P \subset TM \otimes \mathbb{C}$$

that is, a Lagrangian ($rank(P) = n, \omega_{|_P} = 0$), involutive distribution in $TM \otimes \mathbb{C}$.

The Hilbert space of quantum states is then (with some supplementary L^2 conditions)

$$\mathcal{H}_P = \{s \in \Gamma(M, L) : \nabla_{\bar{P}} s = 0\}.$$

Example 1: $M = T^*X \ni (x, p)$ and P is the Schrödinger polarization then $\mathcal{H}_P = L^2(X)$ with quantum sates satisfying $\nabla_{\frac{\partial}{\partial p}} s = 0$.

Example 2: M is Kähler and $P = T^{(1,0)}M$. Then $\mathcal{H}_P = H^0(M, L)$ and quantum states are holomorphic sections, $\nabla_{\frac{\partial}{\partial \bar{z}}} s = 0$. This is the most relevant example for the QHE.

It turns out that quantization is better behaved along families if one includes the half-form correction by taking $L \otimes K^{\frac{1}{2}}$, where K is the canonical bundle. For a surface, a local section of $K^{\frac{1}{2}}$ looks like $\sqrt{dz}$, where z is a local holomorphic coordinate. Including half-forms corresponds to taking spin $\frac{1}{2}$ particles.

Kähler quantization is usually much better behaved that quantization in real polarizations. When $P = \bar{P}$ is a real polarization, sections of L covariantly constant along P must be supported on leaves of P where ∇ has trivial holonomy. These are the Bohr-Sommerfeld leaves (BS leaves).

If M is compact only a finite number of leaves of P will be BS. Thus, sections which are covariantly constant along P will be distributional in nature.

In several interesting cases, the families of Kähler structures $(M, \omega; J_t, g_t)$ approach interesting real polarizations as $t \to +\infty$ and one can describe how holomorphic quantum states localize over BS fibers of the limit polarization.

In the case of the sphere, we will look at such families where the sphere becomes more and more cigar shaped. BS cycles will be specific parallels.

One of the major problems in geometric quantization is the dependence of $\mathcal{H}_P$ on the choice of P. In some cases, if $\mathcal{J}$ is a nice space of Kähler complex structures for (X, ω) one gets a Hilbert bundle

$$\mathcal{H} \to \mathcal{J},$$

with fiber $\mathcal{H}_P$ over $P \in \mathcal{J}$. Of course, one would like to find a natural unitary (projectively) flat connection on $\mathcal{H}$ providing for a unitary identification of quantizations for different polarizations through parallel transport.

A well-known example, still an open problem, is the matter of the unitarity of the KZ/Hitchin connection on the bundle of conformal blocks for Chern-Simons theory with non-abelian gauge groups.

If $\mathcal{J}$ has some (partial) compactification $\bar{\mathcal{J}}$, where $\partial\bar{\mathcal{J}}$ includes some mixed and real polarizations, one would like to have continuous interpolation between quantum states for holomorphic and real polarizations on the boundary.

Sometimes the holomorphic quantum states can be generated by a concrete analytical gadget - called a generalized coherent state transform (CST) - applied to quantum states in a real polarization $P_0 \in \partial\bar{\mathcal{J}}$. The unitarity (or lack of) of this operator then decides the equivalence of quantizations in different polarizations $P, P' \in \bar{\mathcal{J}}$.

Examples include complex Lie groups, complex tori and classical theta functions, non-abelian theta functions on an elliptic curve, symplectic toric manifolds like S^2. The case of classical theta functions corresponds to the unitarity of the KZ connection for Chern-Simons theory with an abelian gauge group on $S \times S^1$.

The CST analytical gadget is intimately related to Hamiltonian dynamics in imaginary time.

The space of Kähler metrics

Let (M, ω, J, γ) be a compact Kähler manifold. Locally, on a sufficiently small open set $U \subset M$, the Kähler form can be written in terms of a (non-unique) Kähler potential

$$\omega = i\partial\bar{\partial}\kappa, \quad \kappa \in \mathbb{C}^{\infty}(U, \mathbb{R}).$$

From the $\partial\bar{\partial}$-lemma, the space of Kähler forms in the class $[\omega] \in H^{1,1}(M)$ is

$$\mathcal{H} = \{\phi \in C^{\infty}(M) : \omega_\phi = \omega + i\partial\bar{\partial}\phi > 0\},$$

that is, two Kähler forms in $[\omega]$ differ by a *global* Kähler potential.

The space of Kähler metrics in the class $[\omega]$ is then given by $\mathcal{H}/\mathbb{R}$.

The Mabuchi metric on $\mathcal{H}$ is

$$||\delta\phi||_\phi^2 = \int_M (\delta\phi)^2 d\mu_\phi, \;\; d\mu_\phi = \frac{1}{n!}\omega_\phi^n.$$

The expression for the curvature of $\mathcal{H}$, as well as other arguments, show that (Donaldson, Semmes), morally,

$$\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega),$$

an infinite-dimensional non-compact symmetric space for the "group" of complexified symplectomorphisms of (M,ω). (This group does not really exist but this is a useful analogy.) This led Donaldson to suggest that geodesics on $\mathcal{H}$ should be generated by "complexified" Hamiltonian flows. Below we will make this suggestion concrete and (more) explicit.

Geodesics on $\mathcal{H}$ are described by the non-linear equation

$$\ddot{\phi} = \frac{1}{2}||\nabla\dot{\phi}||_\phi^2.$$

The (very hard to obtain) analytical and geometrical properties of these geodesics play an important role in recent great developments in Kähler geometry.

Construction of Mabuchi geodesics

Let (M, ω, J, γ) be a compact Kähler manifold and suppose that all the structures (symplectic form ω, complex structure J, Riemannian metric γ) are real analytic.

If X_h is a real analytic Hamiltonian vector field, $h \in C^\omega(M)$, its time t flow, $\varphi_t : M \to M$, will be real analytic in t. Power series (in one variable) have a radius of convergence in the complex plane. This is defined on small open sets on M (Gröbner). Since M is compact there exists some $T > 0$ such that we can analytically continue to complex time τ for $|\tau| < T$.

Let z^j be local holomorphic coordinates on M and consider their (complex) time τ flow

$$z^j_\tau = e^{\tau X_h} z^j = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_h^k(z^j).$$

Whenever it is well defined, the operator $\exp(\tau X_h)$ acts as an automorphism of the algebra of (real analytic) functions:

$$e^{\tau X_h}(fg) = e^{\tau X_h}(f) e^{\tau X_h}(g).$$

Therefore, on overlapping holomorphic coordinate charts the operator $\exp(\tau X_h)$ preserves the (holomorphic) coordinate transformations defining M as a complex manifold.

Theorem: (Mourão-N '15) There exists $T > 0$ such that for $|\tau| < T$ there exists a global complex structure J_τ on M, defined locally by the coordinates z_τ^j, and a unique biholomorphism

$$\varphi_\tau : (M, J_\tau) \to (M, J).$$

We get two equivalent Kähler structures (ie nothing new)

$$(M, \omega, J, \gamma) \cong (M, \varphi_\tau^* \omega, J_\tau, \varphi_\tau^* \gamma_\tau).$$

Since we are taking an Hamiltonian vector field, however, J_τ is still compatible with the original symplectic form ω and we get a new Kähler structure. (Note that $\{z_\tau^i, z_\tau^j\}_{PB} = 0$, since X_h is Hamiltonian.)

Theorem: (Mourão-N '15) For $|\tau| < T$,

i) (M, ω, J_τ) is a Kähler manifold (w/ a new Riemannian metric γ_τ)

ii) There is a reasonably explicit formula for the Kähler potential κ_τ.

Sometimes, these results hold even if M is not compact and for $T = +\infty$.

Theorem: (Mourão-N '15) The family of Kähler metrics γ_τ is a geodesic family with respect to the Mabuchi metric.

Example: The symplectic plane

Consider the symplectic plane $(\mathbb{R}^2, \omega = dx \wedge dp)$ and the Hamiltonian function $h = \frac{1}{2}p^2$, with Hamiltonian vector field $X_h = p\,\partial_x$. We then have in imaginary time $it, t > 0$,

$$z_t = e^{itX_h} x = x + itp.$$

This defines an holomorphic coordinate on the plane for a complex sructure J_t. The Riemannian metric is

$$\gamma_t = t^{-1} dx^2 + t dp^2.$$

Note that, in this case, we can start with the real coordinate x at $t = 0$ and for $t > 0$ we obtain flat Kähler structures on the plane, with the standard Kähler structure corresponding to $t = 1$.

The quantization in the Schrödinger polarization, at $t = 0$, is just the usual

$$L^2(\mathbb{R}, dx) \otimes \sqrt{dx}.$$

At imaginary time $it, t > 0$ we get, with $\nabla = d + ip\,dx$, that quantum states are of the form, with f holomorphic,

$$f(z_t) e^{-\frac{t}{2}p^2} \otimes \sqrt{dz_t} = f(z_t) e^{\frac{(z_t - \bar{z}_t)^2}{8t}} \otimes \sqrt{dz_t},$$

.

These quantizations of the symplectic plane are related by the time t coherent state, or Segal-Bargmann, transform,

$$C_t : \mathcal{H}_{P_0} \;\; \to \;\; \mathcal{H}_{P_t}, \;\; t > 0,$$

which can be described as follows. Let

$$\widehat{h}_{pQ} = i\nabla_{X_h} + h = ip\partial_x - \frac{1}{2}p^2$$

be the Kostant-Souriau prequantum operator for h. Note that

$$\widehat{p}_{pQ} = i\partial_x.$$

We define the quantum operator associacted to $h = \frac{1}{2}p^2$ to be

$$\widehat{h}_Q = \frac{1}{2}\widehat{p}_{pQ}^2 = -\Delta.$$

We then have that the CST for imaginary time $it, t > 0$, can be written as

$$C_t = e^{t\widehat{h}_{pQ}} \circ e^{-t\widehat{h}_Q}$$

The first term, on the left with prequantum operator, plays the role of analytic continuation from x to z_t, and inserts a factor of $\exp(-\frac{1}{2}\kappa)$, where κ is the Kähler potential for (ω, J_t). The term on the right is the heat kernel which is a contraction operator. C_t is the unique (up to phase) unitary operator intertwining $\star$-representations of the Heisenberg group on $\mathcal{H}_{P_0}$ and on $\mathcal{H}_{P_t}$. Note that while x acts by multiplication on $\mathcal{H}_{P_0}$, one has that it is z_t that acts by multiplication on $\mathcal{H}_{P_t}$.

This structure for the evolution of quantum states, under deformations of geometry induced by Hamiltonian flows in imaginary time, vastly generalizes: to complex Lie groups, to classical theta functions on abelian varieties, to non-abelian theta functions at genus one and to symplectic toric manifolds. In general, however, even for the symplectic plane, as soon as h has non-quadratic terms (convexity of h is crucial) the CST is no longer unitary and one does not obtain equivalent quantizations. For symplectic toric manifolds, as the case of the sphere that we will examine next, C_t is not unitary even for quadratic h but it is asymptotically unitary as $t \to +\infty$.

Quantization of S^2 on axially symmetric Kähler structures

Let us take the usual S^1 action on S^2, by rotation around the vertical axis, with symplectic form such that

$$\int_{S^2} \omega = 2\pi N, N \in \mathbb{Z}.$$

Including the half-form correction, the moment polytope is $P = [-\frac{1}{2}, N - \frac{1}{2}]$ and in action-angle coordinates, valid on an open dense subset $\check{P} \times S^1$,

$$\omega = dx \wedge d\theta,$$

where $\check{P}$ is the interior of P. We will consider Mabuchi geodesic families of S^1-invariant Kähler structures described (Guillemin-Abreu) symplectic potentials of the form, for $s > 0$,

$$g_s = \frac{1}{2}\left(x+\frac{1}{2}\right)\log\left(x+\frac{1}{2}\right) + \frac{1}{2}\left(N-\frac{1}{2}-x\right)\log\left(N-\frac{1}{2}-x\right) + \frac{s}{2}x^2.$$

As we will describe, at $s = 0$ this corresponds to the standard Kähler structure on the round sphere and this is a family of Kähler structures corresponding to Hamiltonian evolution in imaginary time $is, s > 0$, for $h = \frac{1}{2}x^2$ where as $s \to +\infty$, S^2 becomes more and more cigar shaped,

$$\gamma_s = g_s^{''} dx^2 + \frac{1}{g_s^{''}} d\theta^2.$$

The holomorphic coordinate on $\check{P} \times S^1$ is then

$$w_s = e^{z_s} = e^{\frac{\partial g_s}{\partial x} + i\theta} = \left(\frac{x - \frac{1}{2}}{N - \frac{1}{2} - x} \right)^{\frac{1}{2}} e^{sx + i\theta}.$$

The connection on the prequantum bundle is $\nabla = d - ixd\theta$ and the half-form corrected sections, which give the 1-particle states for the QHE for the Kähler structure at imaginary time $is, s > 0$, for spin $\frac{1}{2}$ particles, are of the form

$$\sigma_s^m = w_s^m e^{-\kappa_s} \mathbf{1}_P \otimes \sqrt{dz_s}, \quad m = 0, \cdots, N-1,$$

where $\mathbf{1}_P$ is an s-independent unitary section and the Kähler potential at time is is $\kappa_s = \kappa_0 + \frac{s}{2}x^2$ (this is the Legendre transform of g_s).

Note that 1-particle states are labelled by the integral points of P. These correspond to the parallels of S^2 which are Bohr-Sommerfeld.

The prequantum and quantum operators associated to $h = \frac{x^2}{2}$ read

$$\hat{h}_{pQ} = -ix\frac{\partial}{\partial\theta} - \frac{x^2}{2}, \ \ \hat{h}_Q = \frac{1}{2}x_{pQ}^2 = -\frac{1}{2}\frac{\partial^2}{\partial\theta^2}.$$

One obtains that prequantum evolution corresponds to analytic continuation and to the correction of the Kähler potential,

$$e^{s\hat{h}_{pQ}}\sigma_0^m = \sigma_s^m.$$

But, as we have seen the full generalized CST is

$$C_s = e^{s\hat{h}_{pQ}} \circ e^{-s\hat{h}_Q},$$

where the quantum operator part now affects the monomial section σ_0^m with a factor of $e^{-\frac{s}{2}m^2}$.

From results of Baier-Florentino-Kirwin-Mourão-N., we know that, as $s \to +\infty$, the holomorphic polarization associated to J_s converges to the toric polarization generated by $\frac{\partial}{\partial\theta}$ and that, moreover,

$$\lim_{s\to+\infty} C_s \sigma_0^m \sim \delta_m$$

where δ_m is a distributional section supported on the BS cycle $x = m$. C_s is asymptotically unitary as $s \to +\infty$, thanks to the contribution of the quantum operator.

Moreover, we see that the holomorphic monomial sections are eigenvectors of the quantum operator but with m dependent eigenvalue. Thus, acting with the CST in the Laughlin state will give rise to a different evolution than just the evolution corresponding to the analytic continuation given by the prequantum operator.

Application to the QHE on S^2

The evolution of 1-particle states for the quantization of S^2 along the geodesic family of toric invariant Kähler structures described above, via the generalized CST, can now be applied to the fully filled LLL for the integer quantum Hall effect and to the Laughlin states for the FQHE.

Integer QHE on S^2

The LLL for the IQHE for the round sphere ($s = 0$) is described by the totally antisymmetric product of 1-particle states

$$\Psi_0^{IQHE} = \sum_{\tau \in S_N} sgn(\tau) \sigma_0^{\tau_1}((w_0)_1) \otimes \cdots \otimes \sigma_0^{\tau_N}((w_0)_N),$$

which is a section of $L^{\boxtimes N} = \pi_1^* L \otimes \cdots \otimes \pi_N^* L \to S^2 \times \cdots \times S^2$ where $\pi_j : S^2 \times \cdots \times S^2 \to S^2$ is the projection onto the jth factor, $j = 1, \ldots N$.

The generalized CST acts on this state in a natural way,

$$C_s \Psi_0^{IQHE} = e^{-\frac{s}{2} \sum_{m=0}^{N-1} m^2} \sum_{\tau \in S_N} sgn(\tau) \sigma_s^{\tau_1}((w_s)_1) \otimes \cdots \otimes \sigma_s^{\tau_N}((w_s)_N),$$

and the overall factor can be absorbed in the normalization of the LLL state.

It is interesting to note that for very large deformations $s \to +\infty$ there will be localization along the BS cycles.

Fractional QHE on S^2

For the fractional QHE with filling fraction $\nu = N_e/N = 1/k$, with odd k, we have the known Laughlin state for the round sphere ($s = 0$),

$$\Psi_{Laughlin} = \Pi_{1\leq i<j\leq N_e}((w_0)_i - (w_0)_j)^k \, e^{-\sum_{j=0}^{N_e} \kappa_0((w_0)_j)}.$$

This state is an antisymmetric section of $L^{\boxtimes N_e}$ and (including the half-form correction) it is convenient to expand it in terms of an eigenbasis of the quantum operator given by Slater determinants

$$\psi_s^\lambda = \sigma_s^{\lambda_1} \wedge \cdots \wedge \sigma_s^{\lambda_{N_e}},$$

where $\lambda = (\lambda_1, \ldots, \lambda_{N_e})$ is such that $0 \leq \lambda_1 < \cdots < \lambda_{N_e} \leq N - 1$.

We have

$$C_s \psi_0^\lambda = e^{-\frac{s}{2}\sum_{j=1}^{N_e} \lambda_j^2} \psi_s^\lambda$$

Dunne ('93) has given a decomposition of the Laughlin state in terms of this basis

$$\Psi_{Laughlin}^{s=0} = \sum_\lambda a_\lambda(k) \psi_0^\lambda.$$

We then obtain the evolution of the Laughlin state under the generalized CST

$$C_s \Psi^{s=0}_{Laughlin} = \sum_\lambda a_\lambda(k) e^{-\frac{s}{2}\sum_{j=1}^{N_e}\lambda_j^2} \psi_s^\lambda,$$

where different Slater terms evolve differently due to the contribution of the quantum operator. Of course, this will also have an effect on the the particle density

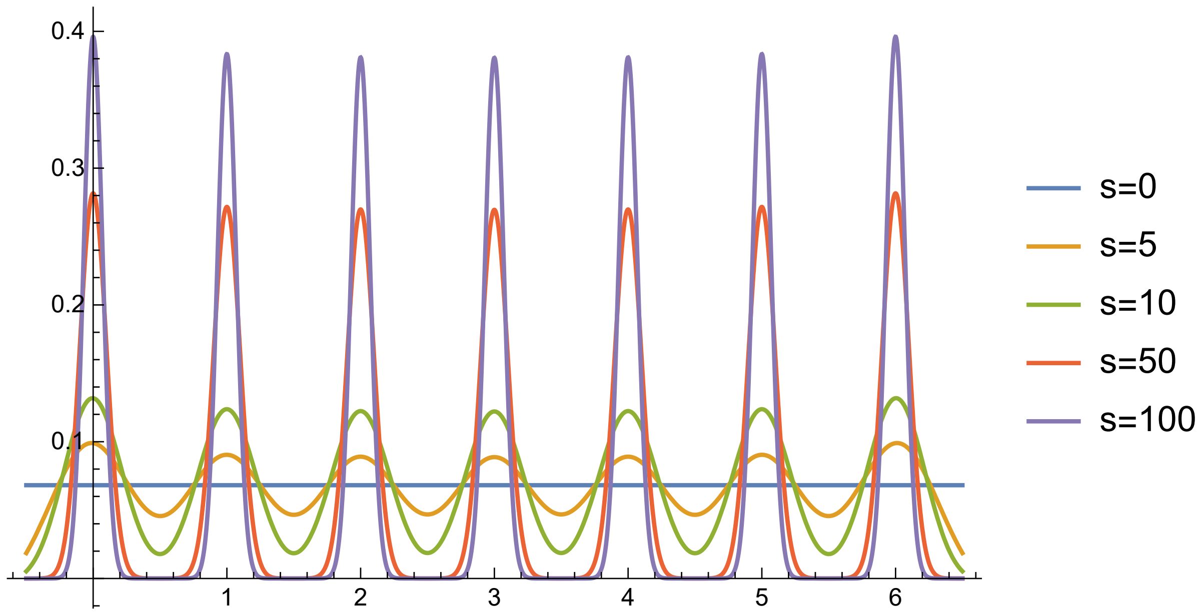
$$\rho_s(x) = \frac{\langle C_s \Psi^{s=0}_{Laughlin}, \sum_{j=1}^{N_e} \delta(x - x_j) C_s \Psi^{s=0}_{Laughlin} \rangle}{||C_s \Psi^{s=0}_{Laughlin}||^2}.$$

If the quantum operator is not taken into account, as $s \to +\infty$ one gets a dominant contribution from the Slater with largest $|\lambda|^2$. Including the quantum operator, however, gives a particle density which is more evenly distributed among BS cycles.

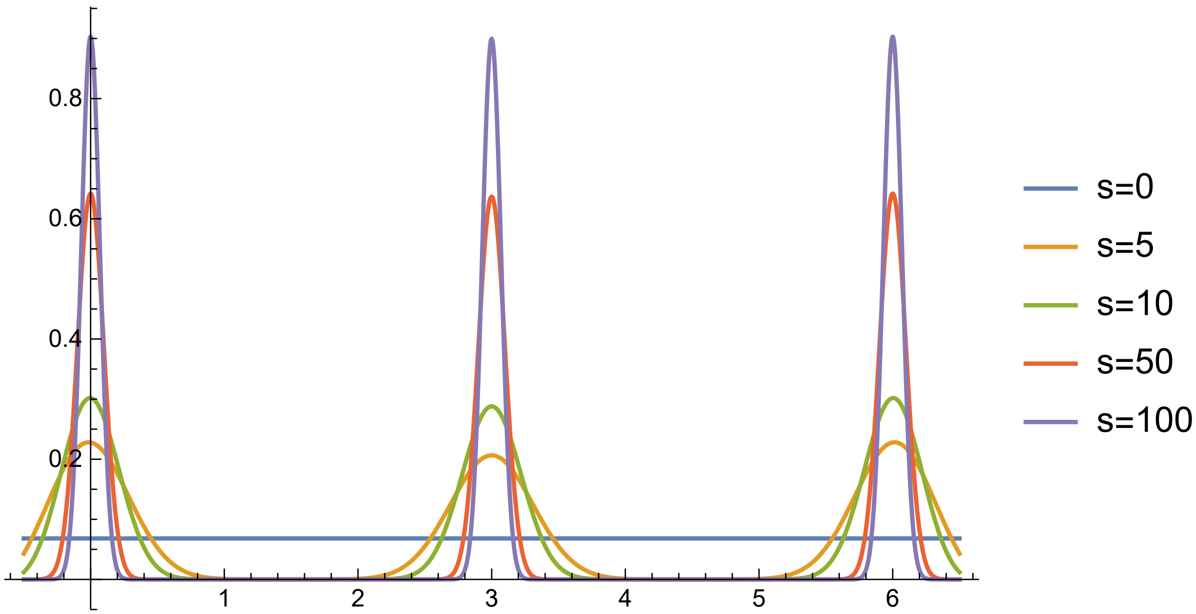
Note that taking $N \to +\infty$, ie $P = [-\frac{1}{2}, +\infty)$, in the above describes rotationally invariant non-flat Kähler structures on the plane generated by the Hamiltonian flow in imaginary time of $h = \frac{1}{2}(x_1^2 + x_2^2)$.

Illustrative examples for $k = 3$

Density profiles of the evolved states of 3 particles on the sphere with $s = 0$, $s = 5$, $s = 10$, $s = 50$ and $s = 100$, for $h = \frac{1}{2}x^2$:



Density profiles of states of 3 particles on the sphere evolved only with pre-quantum operator for $s = 0$, $s = 5$, $s = 10$, $s = 50$ and $s = 100$, for $h = \frac{1}{2}x^2$:



Final conclusions

Hamiltonian flows in imaginary time give a very useful tool for the study of the evolution of 1-particle states for the QHE under the corresponding deformations of geometry.

In geometric quantization, the natural law of evolution of quantum states under these deformations of the geometry is given by a generarized CST. Besides the prequantum evolution, which corresponds to analytic continuation, there is also evolution under a quantum operator which affects different monomial 1-particle states on S^2 in different ways. Thus, the CST predicts an evolution of the Laughlin state which is not just given by analytic continuation.

For extremely deformed axially symmetric geometries on S^2, where the sphere becomes more and more cigar shaped, 1-particle states concentrate along BS cycles and this affects the particle density for the Laughlin state.

Note that for the QHE for flat geometries on the torus, labelled by a modular parameter $\tau = \tau_1 + i\tau_2, \tau_2 > 0$, the CST evolution gives results which are identical to the known results for the Laughlin states on flat tori: in this case, if one includes the half-form correction (that is, for spin $\frac{1}{2}$ particles), the CST evolution just maps - unitarily - between theta functions for different $\tau's$. In this case, the CST is the (unique up to phase) unitary intertwiner between the (unique up to isomorphism) irreducible representation of the finite Heisenberg group. This is not true if the quantum operator is not included.

THANK YOU.