Equipartition of Entanglement in Quantum Hall States

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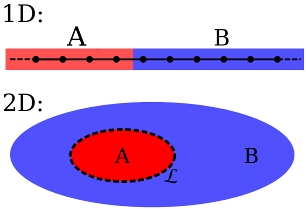
Motivations

Entanglement and symmetries

- $|\Psi\rangle$ many-body quantum state and a bipartition $A+B$
- $U(1)$ charge $\hat{Q} = \hat{Q}_A + \hat{Q}_B$
- $\hat{Q}\,|\Psi\rangle = q_t\,|\Psi\rangle$
- Reduced density matrix $\rho_A = \mathrm{Tr}_B\,(|\Psi\rangle\langle\Psi|)$, $\left[\rho_A, \hat{Q}_A\right] = 0$
- Block diagonal structure: $\rho_A = \bigoplus_q p_q \rho_A(q)$, $\mathrm{Tr}_A\,(\rho_A(q)) = 1$



- p_q: Full counting statistics (FCS) for the charge, an experimentally relevant quantity.
- $\rho_A(q)$: provides a better insight on the entanglement structure of $|\Psi\rangle$.

Equipartition of entanglement

Defining a Von Neumann entanglement entropy per q sector

$$S_1 = -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right) = -\sum_q p_q \log p_q + \sum_q p_q S_1(q)$$

- $-\sum_q p_q \log p_q$: Shannon entropy for the FCS.
- $S_n(q) = \frac{1}{1-n} \log\left(\mathrm{Tr}_A \rho_A^n(q)\right)$: Symmetry resolved Renyi entropy.

Equipartition of entanglement entropy (J. Xavier, F. Alcaraz and G. Sierra PRB (2018)):

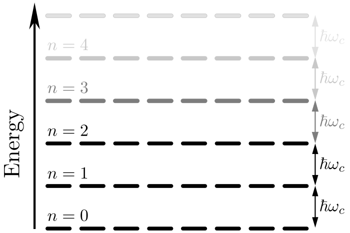
$S_n(q)$ *does not depend on* q (in the thermo. limit, $A+B \to \infty$, $A \to \infty$), $S_1(q) = S_1 + \sum_q p_q \log p_q$

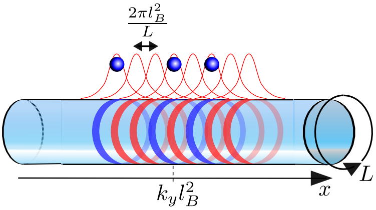
Valid for critical 1d systems, integrable models, topological phases,...

Integer quantum Hall effect

Quantum Hall effect

Landau levels (spinless case)





- Cyclotron frequency : $\omega_c = \frac{eB}{m}$,
- Filling factor : $\nu = \frac{hn}{eB} = \frac{N}{N_\Phi}$
- Partial filling + interaction $\rightarrow$ FQHE
- Lowest Landau level $(\nu < 1)$:
  $z^m \exp\left(-|z|^2/(4l_B^2)\right)$
- N-body wave function :
  $\Psi = P(z_1, ..., z_N) \exp(-\sum |z_i|^2/(4l_B^2))$
- Landau gauge and cylinder : ring-like orbital centered around $k_m l_B^2$, k_m quantized.

Evaluating the symmetry-resolved EE

Generating function: $\hat{Z}_n(\alpha) = \mathrm{Tr}_A \left(e^{i\alpha \hat{Q}_A} \rho_A \right)$.

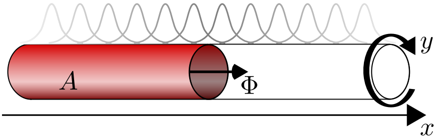
$$Z_n(q) \equiv \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} \hat{Z}_n(\alpha) = \sum_q \mathrm{Tr}_A \left(p_q^n \rho_A^n(q) \right)$$

- FCS: $p_q = Z_1(q)$
- Sym-res EE: $S_n(q) = \frac{1}{1-n} \log \left(\frac{Z_n(q)}{Z_1^n(q)} \right)$, $S_1(q) = -\frac{d}{dn} \frac{Z_n(q)}{Z_1^n(q)} \bigg|_{n=1}$

For free fermions in Slater $|\Omega\rangle$: evaluation from the correlation matrix $C(r, r') = \langle \Omega | \Psi^\dagger(r') \Psi(r) | \Omega \rangle$. If C_A is its restriction to A, with eigenvalues λ_m:

$$\hat{Z}_n(\alpha) = \prod_{m \in \mathbb{Z}} \left(\lambda_m^n e^{i\alpha(1-\lambda_m)} + (1-\lambda_m)^n e^{-i\alpha \lambda_m} \right)$$

Case of the filled LLL



- An extra knob: Φ flux along the cylinder axis.
- A region semi-infinite cylinder ($x < 0$).

Wavefunction in the LLL:

$$\phi_{k_m}(x, y) = \frac{1}{\sqrt{L\sqrt{\pi}}} e^{ik_m y} e^{-(x-k_m)^2/2}, \quad k_m \in \frac{2\pi}{L}(\mathbb{Z} + \Phi)$$

- $\lambda_m = \frac{1}{2}(1 - \mathrm{erf}(k_m))$, overlap of orbital m on A (the overlap matrix have the same non-zero eigenvalues than C_A).
- Defining $\langle : \hat{Q}_A : \rangle$ properly: $\frac{1}{2} - \Phi$ (up to gaussian corrrections). The orbital at $m = 0$ is cut in the middle at $x = 0$ (for $\Phi = 0$).

FCS and Symmetry-resolved EE

For the IQHE, both can be derived exactly

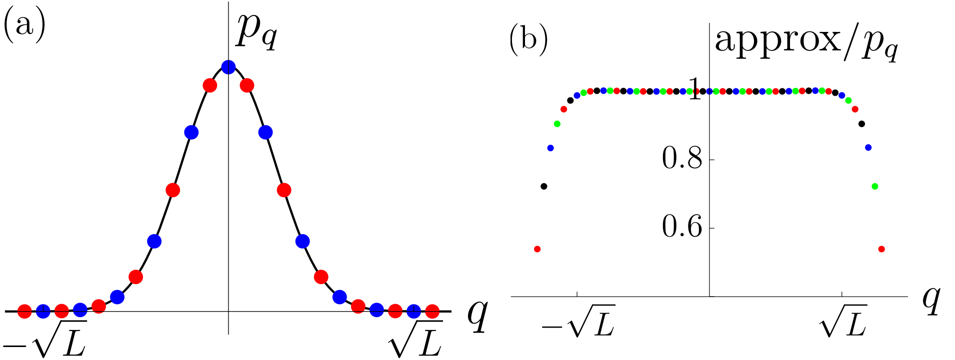
$$p_q \propto \frac{e^{-\frac{q^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}, \quad \sigma^2 = \frac{L}{(2\pi)^{3/2}}$$

$$S_n(q) \sim S_n - \tfrac{1}{2}\log L + A_n - B_n \frac{q^2}{L} + C_n \frac{q^4}{L^3}$$

$$A_n = \alpha_n + \frac{\alpha_n'}{L}, \quad B_n = \beta_n + \tfrac{\beta_n'}{L}, \quad C_n = \mathrm{constant}$$

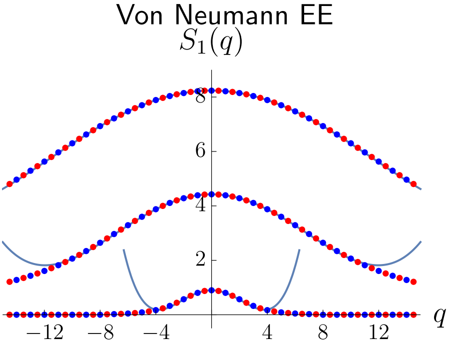
At large L and small q deviations, *equipartition of EE*:
$S_n(q) \simeq S_n - \frac{1}{2}\log L + \alpha_n$
$\log L$ correction compared to the EE ($S_n = c_n L - \gamma$, $\gamma = 0$ for filled LLL).

Numerical results: FCS



- $L = 25$, blue dots $\Phi = 0$, red dots $\Phi = 1/2$ (Φ turns q into a continuous variable).
- Gaussian approximation perfectly captures the FCS for small fluctuations ($\sim \sqrt{L}$).

Numerical results: Symmetry-resolved EE



- $L = 10, 30, 50$ (from bottom to top), $\Phi = 0$ and $\Phi = 1/2$.
- Very good agreement with the analytical expression (continuous blue line, no fit needed).

Fractional quantum Hall effect

The Laughlin wave function

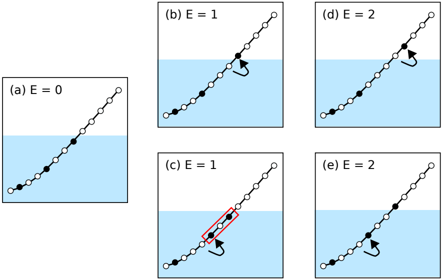
A (very) good approximation of the ground state at $\nu = \frac{1}{p}$

$$\Psi_L(z_1, ...z_N) = \prod_{i<j}(z_i - z_j)^p e^{-\sum_i \frac{|z_i|^2}{4l^2}}$$

Excitations with fractional charge $\frac{e}{p}$ and fractional statistics

Edge excitations

- A chiral $U(1)$ boson with a dispersion relation $E \simeq \frac{2\pi v}{L} n$
- The degeneracy of each energy level is given by the sequence $1, 1, 2, 3, \ldots$



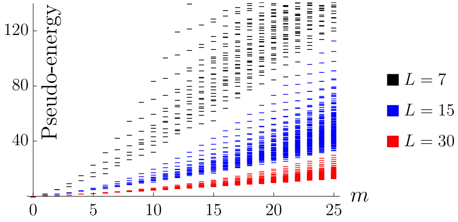
Li-Haldane conjecture

- Rewriting $\rho_A = e^{-H_{\mathrm{ent}}}$, where H_{ent} is the entanglement Hamiltonian.
- For topological phases, H_{ent} is the edge/surface mode hamiltonian H_{edge}.
- For FQH states captured by a CFT: $H_{\mathrm{edge}} = \frac{2\pi v}{L}\left(L_0 - \frac{c}{24}\right)$, L_0 zero mode of the stress tensor and c central charge.
- For Laughlin state $\nu = 1/p$: $c = 1$, $L_0 = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n$.
- a_n bosonic mode of the $U(1)$ chiral current, $a_0 = \sqrt{p}\hat{Q}_A$.

$$p_q \propto \frac{e^{-\frac{q^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}, \quad \sigma^2 = \frac{L}{2\pi p v}, \quad q = \frac{\delta}{p} + \mathbb{Z}$$

$\delta = 0, ..., p-1$ is the topological sector.

Corrections to the Li-Haldane conjecture



m momentum, filled LLL

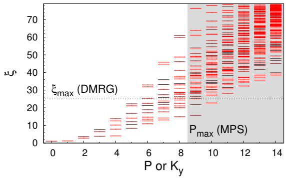
- The actual spectrum of ρ_A is not linear (at finite L).
- Corrections to the CFT spectrum (Dubail et al., PRB (2012)) by adding *irrelevant local perturbations*.

$$H_{\text{ent}} = \frac{2\pi v}{L}\left(L_0 - \frac{c}{24}\right) + \sum_j g_j \left(\frac{\pi}{L}\right)^{\delta_j - 1} V_j$$

- V_j zero mode of Φ_j local fields with dimension $\Delta_j > 2$.
- g_j are not universal and should be fine-tuned but can be computed *exactly* for the filled LLL ($p = 1$).
- $p > 1$ leads to the same expression for $S_n(q)$ but A_n, B_n and C_n should be fitted.

Matrix Product States for FQH states

$$|\Psi\rangle = \sum_{\{m_i\}} \langle \alpha_L | A^{[m_1]} \ldots A^{[m_{N_{\rm orb}}]} | \alpha_R \rangle \; |m_1, \ldots, m_{N_{\rm orb}}\rangle$$



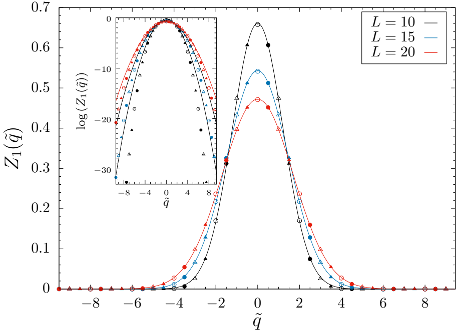
$\{A^{[m]}\}$ is a set of matrices + (α_l, α_r) boundary conditions

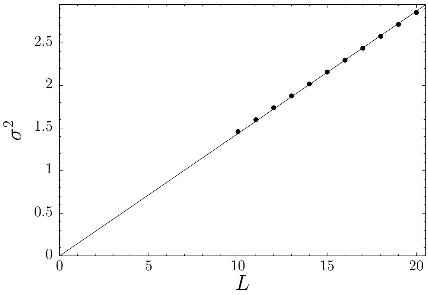
The $A^{[m]}_{\alpha,\beta}$ matrices have two types of indices

- $[m]$ is the physical index (for FQH, the orbital occupation)
- (α, β) are the auxiliary space indices $1, \ldots, \chi$.
- Known exactly for the Laughlin states, Zaletel and Mong, PRB (2012)
- The auxiliary space is the CFT Hilbert space of the edge mode..
- Truncated CFT: fix a maximum conformal dimension P_{max}.
- In general χ is of the order of $\exp \mathcal{S}_A$ ($\mathcal{S}_A$ is the entanglement entropy).

Numerical results: FCS

We focus on the bosonic Laughlin state $\nu = \frac{1}{2}$.

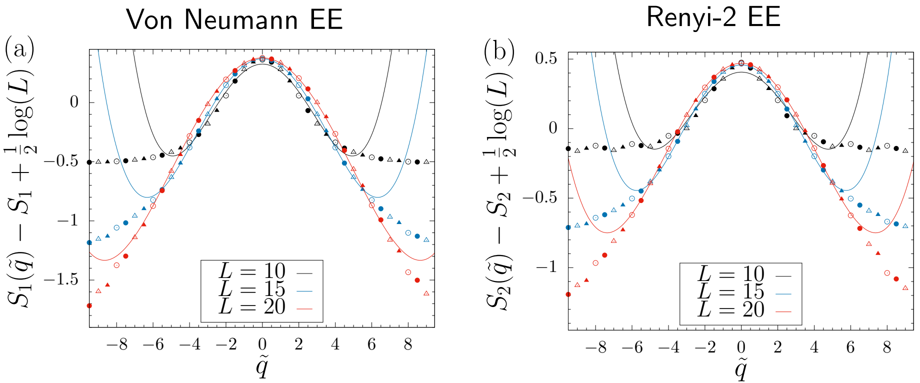




$$p_q \sim \frac{e^{-\frac{q^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \quad \text{and} \quad \sigma^2 = \frac{L}{2\pi p v}$$

Here $v = 0.144(1)$.

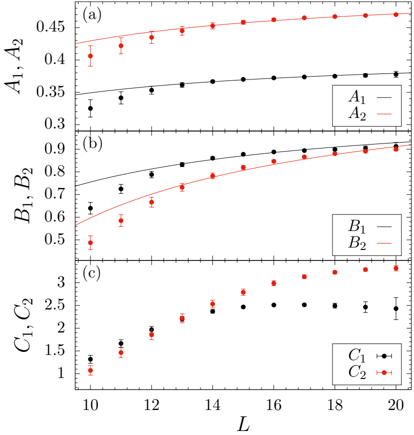
Numerical results: Entanglement Entropies



Technical challenges:

- Innitely long cylinder but limited in perimeter L (require large $P_{\max}$ and thus matrices).
- Focusing on subdominant terms is usually hard (more prone to finite size effects).

Numerical results: corrections



- From Li-Haldane+perturbations:
  - $A_n = \alpha_n + \frac{\alpha_n'}{L}$
  - $B_n = \beta_n + \frac{\beta_n'}{L}$
  - $C_n = \text{constant}$
- Fairly good agreement for A_n and B_n.
- Still observe finite size effects for C_n.

FQH Laughlin states do satisfy the equipartition of EE.

Conclusion

- Symmetry-resolved EE provides a finer perspective on entanglement.
- Quantum Hall effect is a nice playground to test new concept: analytical results for IQHE and analytical/numerical results for FQHE.
- *Both IQHE and the FQH Laughlin states satisfy the equipartition of entanglement entropy.*
- Outlook:
    - Non-abelian states (Moore-Read) and equipartition of EE?
    - Any violation of EE equipartition in QHE?