Effective models for emerging anyons

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Works with Michele Correggi, Romain Duboscq, Théotime Girardot, Antoine Levitt, Douglas Lundholm

Many-body quantum mechanics

- Many-body classical mechanics: N indistinguishable particles with positions $x_1, \dots, x_N \in \mathbb{R}^d$, momenta $p_1, \dots, p_N \in \mathbb{R}^d$
- Probability measure invariant under label exchanges

$$\mu_N(x_1, \dots, x_N; p_1, \dots, p_N)$$

Kinetic energy $= \sum_{j=1}^N |p_j|^2$

- Many-body quantum mechanics (Heisenberg uncertainty): wave-function $\Psi_N \in L^2(\mathbb{R}^{dN}, \mathbb{C})$ with

$$|\Psi_N(x_1, \dots, x_N)|^2 \leftrightarrow \int_{\mathbb{R}^{dN}} \mu_N(x_1, \dots, x_N; p_1, \dots, p_N) dp_1 \dots dp_N$$

$$|\widehat{\Psi_N}(p_1, \dots, p_N)|^2 \leftrightarrow \int_{\mathbb{R}^{dN}} \mu_N(x_1, \dots, x_N; p_1, \dots, p_N) dx_1 \dots dx_N$$

Kinetic energy $= \sum_{j=1}^N \left(-\mathrm{i}\nabla_{x_j}\right)^2 = \sum_{j=1}^N -\Delta_{x_j}$

- $|\Psi_N|^2$ and $|\widehat{\Psi_N}|^2$ must be invariant under label exchanges
- "Most natural" possibilities for quantum statistics of particles

$\Psi_N(x_1, \dots, x_i, \dots, x_j, \dots, x_N) = \Psi_N(x_1, \dots, x_j, \dots, x_i, \dots, x_N)$ **Bosons**

$\Psi_N(x_1, \dots, x_i, \dots, x_j, \dots, x_N) = -\Psi_N(x_1, \dots, x_j, \dots, x_i, \dots, x_N)$ **Fermions**

Exchanges and statistics

Why are there only bosons and fermions ?

Girardeau 65, Souriau 67, Laidlaw-Morette de Witt 71, Leinaas-Myrheim 77, Goldin-Menikoff-Sharp 81, Wilczek 82 ...

- Wave-function Ψ_N on configuration space including indistinguishability

$$C_N^d = \left(\mathbb{R}^{dN} \setminus \{x_i = x_j \text{ for some } i \neq j\}\right) / \,\text{Label permutations}$$

- Exchange = physical process = closed loop γ in C_N^d
- Exchange phase $\alpha(\gamma) \in [0, 2[$, only depends on γ modulo continuous deformations (gauge argument)
- $\gamma \mapsto e^{\mathrm{i}\pi\alpha(\gamma)}$ representation of fundamental homotopy goup $\pi^1(C_N^d)$ of C_N^d

In 3D, $\pi^1(C_N^3)$ = permutation group, only trivial and sign representations, bosons and fermions !

- BUT in 2D, the fundamental group of C_N^2 is the braid group.
- Has a continuum of 1D representations, indexed by $\alpha \in [0, 2[$

In a 2D universe, exchange phase of general $e^{\mathrm{i}\pi\alpha}$ is a logical possibility: any-ons !

Anyon gauge and magnetic gauge

- Consider seriously the possibility that some $\Phi_N \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ satisfies

$$\Phi_N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = e^{\mathrm{i}\pi\alpha}\Phi_N(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N)$$

  for some $\alpha \neq 0, 1$. Φ_N would be multi-valued, not a function ...
- Way out: introduce a bosonic (symmetric) Ψ_N such that

$$\Phi_N = \prod_{1 \leq j < k \leq N} e^{\mathrm{i}\alpha \,\mathrm{angle}(x_j - x_k)} \Psi_N$$

- Quantum kinetic energy $p_{x_j}^2 = (-\mathrm{i}\nabla_{x_j})^2 = -\Delta_{x_j}$

$$\boxed{\left\langle \Phi_N \middle| \sum_{j=1}^{N} -\Delta_{x_j} \middle| \Phi_N \right\rangle_{L^2} = \left\langle \Psi_N \middle| \sum_{j=1}^{N} \left(-\mathrm{i}\nabla_{x_j} + \alpha A_{\mathrm{any}}(x_j)\right)^2 \middle| \Psi_N \right\rangle_{L^2}}$$

- Magnetic vector potential

$$A_{\mathrm{any}}(x_j) = \sum_{k \neq j} \frac{(x_j - x_k)^{\perp}}{|x_j - x_j|^2}$$

- Magnetic field

$$B_{\mathrm{any}} = \mathrm{curl}\, A_{\mathrm{any}} = 2\pi \sum_{k \neq j} \delta_{x_j = x_k}$$

Anyon statistics $\Leftrightarrow$ Interactions via magnetic charges (thin solenoids)

Emerging anyons

- World is 3D (at least ...) $\Rightarrow$ fundamental particles are bosons or fermions.
- But a quasi-particle of a 2D system is a genuinely 2D object ...

Fractional quantum Hall physics

- Electrons trapped in 2D (semi-conductors hetero-junctions).
- Very large perpendicular magnetic field: quenched kinetic energy.
- Repulsive Coulomb interactions $\rightsquigarrow$ Laughlin's state 1983
- Elementary excitations thereof behave as anyons (Halperin 84, Arovas-Schrieffer-Wilczek 84)
- Hallmark experiments: Bartolomei-et al-Fève, Nakamura-et al-Manfra 20

A thought experiment in FQH physics

- Two species of 2D particles (bosons or fermions): bath $X_N = (x_1, \dots, x_N)$ tracers $Y_M = (y_1, \dots, y_M)$
- Large magnetic field B : bath minimizes its magnetic kinetic energy
- Intra-species repulsion: bath particles avoid one another
- Inter-species repulsion: tracers avoid bath particles
- Ansatz for the joint wave-function of the system, with $n \geq 2$ integer

$$\Psi_{N+M}(X_N; Y_M) = C_{\mathrm{qh}}(Y_M) \prod_{1 \leq j < k \leq N} (z_i - z_j)^n e^{-\frac{B}{4}\sum_{j=1}^N |z_j|^2} \prod_{j=1}^N \prod_{k=1}^M (z_j - \zeta_k)\, \Phi_M(Y_M)$$

with $\mathbb{R}^2 \ni x_j \leftrightarrow z_j \in \mathbb{C}$ and $\mathbb{R}^2 \ni y_k \leftrightarrow \zeta_k \in \mathbb{C}$

Statement (Lundholm-Rougerie 16)

$$\langle \Psi_{N+M} | -\Delta_{y_1} | \Psi_{N+M} \rangle \approx \langle \Phi_M | H_{\mathrm{eff}} | \Phi_M \rangle$$

with an effective anyonic Hamiltonian acting only on tracers

$$\boxed{H_{\mathrm{eff}} = \left(-\mathrm{i}\nabla_{y_1} + \frac{1}{n} B \frac{y_j^\perp}{2} - \frac{1}{n} A_{\mathrm{any}}(y_1) \right)^2}$$

(also contains a uniform "external" magnetic field)

Ideal 2D anyons

- On $L^2_{\rm sym}(\mathbb{R}^{2N})$ or on $L^2_{\rm asym}(\mathbb{R}^{2N})$, study

$$H_N^\alpha = \sum_{j=1}^N \left(-\mathrm{i}\nabla_{x_j} + \alpha A_{\rm any}(x_j)\right)^2$$

- A many-body problem, with singular gauge potential

$$A_{\rm any}(x_j) = \sum_{k\neq j} \frac{(x_j - x_k)^\perp}{|x_j - x_j|^2}$$

- Very few exact results, e.g. LLL anyons (Ouvry et al)
- Self-adjointness (Adami, Teta, Correggi, Oddis, Fermi ...)
- Pauli principle (Lundholm, Solovej, Larson, Seiringer ...)
- $\leftrightarrow$ Lieb-Thirring inequality. For any α (except $\alpha = 0$ on $L^2_{\rm sym}$)

$$\boxed{\exists \text{ universal constant } C_\alpha,\ \langle \Psi_N | H_N^\alpha \Psi_N\rangle \geq C_\alpha |\Omega|^{-1} N^2}$$

if all particles are inside bounded $\Omega \subset \mathbb{R}^2$ and $\int_{\Omega^N} |\Psi_N|^2 = 1$

Average-field approximation and scaling limits

- Tame the singularity with a mollifier at scale R

$$H_N^{\alpha,R} = \sum_{j=1}^{N} \left(-\mathrm{i}\nabla_{x_j} + \alpha \sum_{k\neq j} \nabla^\perp w_R(x_j - x_k) \right)^2$$

$$= \sum_{j=1}^{N} -\Delta_{x_j} + \alpha \sum_{k\neq j} \Big((-\mathrm{i}\nabla_{x_j}) \cdot \nabla^\perp w_R(x_j - x_k)) + \mathrm{h.c.} \Big)$$

$$+ \alpha^2 \sum_{\ell\neq k\neq j} \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_\ell)$$

$$+ \alpha^2 \sum_{k\neq j} \left| \nabla^\perp w_R(x_j - x_k) \right|^2$$

- Almost bosonic limit: act on L^2_{sym}, expect $\Psi_N \approx u^{\otimes N}$

$$N \to \infty, \quad N\alpha = \beta \text{ fixed}, \quad R = N^{-\eta}, \quad 0 < \eta < \frac{1}{4}$$

- Almost fermionic limit: act on L^2_{asym}, expect $\Psi_N(x_1, \ldots, x_N) \approx \det(u_j(x_k))$

$$N \to \infty, \quad \sqrt{N}\alpha = \beta \text{ fixed}, \quad R = N^{-\eta}, \quad 0 < \eta < \frac{1}{4}$$

Almost bosonic limit

- Boson-based anyons in a trap. On $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$:

$$H_N^\alpha = \sum_{j=1}^{N} \left(-\mathrm{i}\nabla_{x_j} + \alpha \sum_{k \neq j} \frac{(x_j - x_k)^\perp}{|x_j - x_k|^2} \right)^2 + V(x_j)$$

- Simplest question = ground state, lowest eigenfunction

$$E(N,\alpha) = \inf \left\{ \langle \Psi_N | H_N^\alpha | \Psi_N \rangle_{L^2}, \Psi_N \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N}), \int |\Psi_N|^2 = 1 \right\}$$

- Bosons with pair interactions w: ansatz $\Psi_N = u^{\otimes N}$ leads to non-linear Schrödinger / Gross-Pitaevskii

$$\int_{\mathbb{R}^d} \left| \left(-\mathrm{i}\nabla + \frac{B}{2} x_j^\perp \right) u \right|^2 + V|u|^2 + \frac{1}{2} \left(w * |u|^2 \right) |u|^2$$

- Approximate model for "almost bosonic anyons"

$$\boxed{\mathcal{E}_\beta^{\mathrm{af}}[u] = \int_{\mathbb{R}^2} \left| \left(-\mathrm{i}\nabla + \beta A_{\mathrm{any}}[|u|^2] \right) u \right|^2 + V|u|^2, \quad A_{\mathrm{any}}[|u|^2] = |u|^2 * \frac{x^\perp}{|x|^2}}$$

- Self-consistent magnetic field $\rightsquigarrow$ Quasi-linear/non-local problem

Average field approximation

- Use a mollifier of radius R to regularize the anyon field $A_{\mathrm{any}} \rightsquigarrow A_{\mathrm{any}} * \chi_R$

$$H_N^{\alpha,R} = \sum_{j=1}^{N} \left(-\mathrm{i}\nabla_{x_j} + \alpha \sum_{k \neq j} A_{\mathrm{any}}^R(x_j) \right)^2 + V(x_j)$$

- Use a trapping potential

$$V(x) \geq c|x|^s - C \text{ with } s > 0$$

Theorem (Girardot 19, Lundholm-Rougerie 15)

Scaling: with $\beta \in \mathbb{R}$ and $0 < \eta < 1/4$ fixed

$$\alpha = \frac{\beta}{N}, \quad R = N^{-\eta}$$

(i) Lowest eigenvalue satisfies

$$\boxed{\frac{E(N,\alpha,R)}{N} \to E_\beta^{\mathrm{af}}}$$

the minimum of $\mathcal{E}_\beta^{\mathrm{af}}$ under unit mass constraint.
(ii) Consequences for ground states: convergence of reduced density matrices.

Numerics

$$\mathcal{E}_\beta^{\mathrm{af}}[u] = \int_{\mathbb{R}^2} \left| \left(-\mathrm{i}\nabla + \beta A_{\mathrm{any}}[|u|^2] \right) u \right|^2 + V|u|^2, \quad A_{\mathrm{any}}[|u|^2] = |u|^2 * \frac{x^\perp}{|x|^2}$$

- Numerics by Romain Duboscq (inspired by GPE-Lab, Duboscq-Antoine)
- Pre-conditioned conjugated gradient method
- Spectral method: FFT with a Cartesian grid

What to expect ?

Cf vortices in Ginzburg-Landau/Gross-Pitaevskii theory

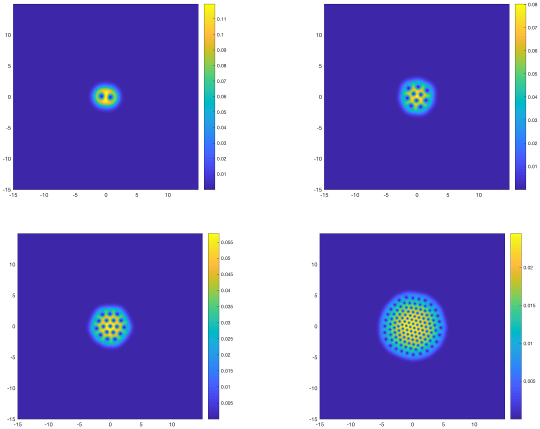
- Theorems on the $\beta \to \infty$ limit (Correggi-Lundholm-Rougerie 17). Energy and density found by minimizing

$$\mathcal{E}_\beta^{\mathrm{TF}}[\rho] = \int_{\mathbb{R}^2} \beta e(1,1) \rho(x)^2 + V(x)\rho(x), \quad \rho \geq 0, \quad \int_{\mathbb{R}^2} \rho = 1$$

- $e(1,1)$ is an unknown parameter (energy density in the thermodynamic limit at $\beta = 1$ and density $= 1$)
- Proof suggests nucleation of vortices : phase singularities/zeroes
- Vortices create circulation to compensate the self-consistent magnetic field
- Vortex density should be proportional to matter density $|u|^2$

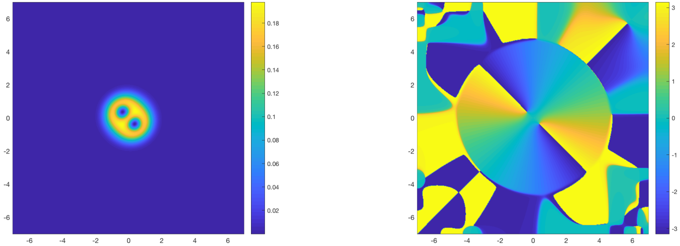
Quadratic trap, $V(x) = |x|^2$

(Most plots to be found in Correggi-Duboscq-Lundholm-Rougerie 19)

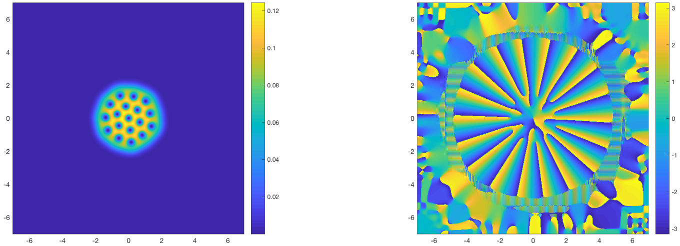


Density $|u|^2$, $\beta = 5, 15, 25, 140.$

Quartic trap, $V(x) = |x|^4$

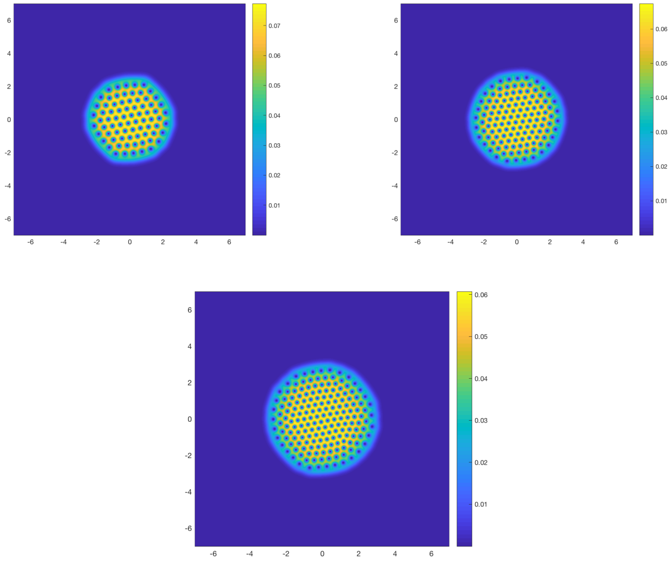


$\beta = 5$, density $|u|^2$, phase $\arg\left(\frac{u}{|u|}\right)$.



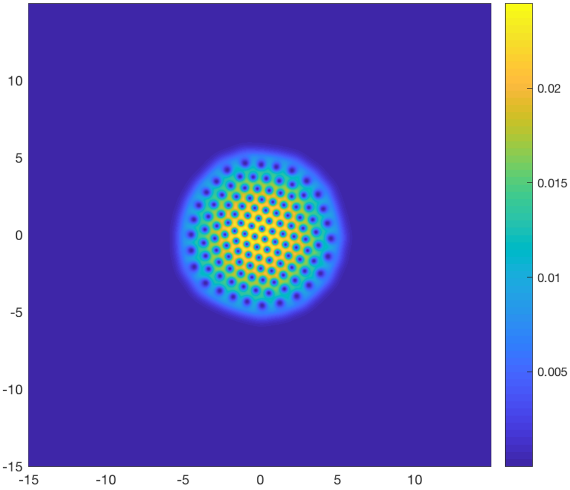
$\beta = 25$, density, phase.

Quartic trap, $V(x) = |x|^4$



Density, $\beta = 90, 140, 195$.

Inhomogeneous vorticity



$$V(x) = |x|^2, \beta = 140$$

Almost fermionic anyons

- Action on $L^2_{\mathrm{asym}}(\mathbb{R}^{2N})$ of

$$H_N^{\alpha,R} = \sum_{j=1}^N \left(-i\hbar\nabla_{x_j} + \alpha \sum_{k\neq j} \nabla^\perp w_R(x_j - x_k) \right)^2 + V(x_j)$$

- Extension parameter $R > 0$

$$\nabla^\perp w_R(x) = \frac{1}{R^2} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \chi\left(\frac{y}{R}\right) dy$$

- Trapping potential

$$V(x) \geq c|x|^s - C \text{ with } s > 0$$

- $\sqrt{N}\alpha$ and $\hbar$ fixed $\Leftrightarrow$ semi-classical scaling

$$N \to \infty, \quad \hbar = N^{-1/2}, \quad N\alpha = \beta \text{ fixed}, \quad R = N^{-\eta}, \quad 0 < \eta < \frac{1}{4}$$

- Ground state energy and associated minimizers

$$E(N,R,\alpha) := \inf \left\{ \left\langle \Psi_N | H_N^{\alpha,R} \Psi_N \right\rangle_{L^2}, \Psi_N \text{ fermionic}, \int_{\mathbb{R}^{2N}} |\Psi_N|^2 = 1 \right\}$$

- Reminiscent of fermionic mean-field limits (Lieb-Simon, Thirring, Lieb-Solovej-Yngvason, Fournais-Lewin-Solovej, Fournais-Madsen ...)

Semi-classical/mean-field energy

- A positive measure $m(x, p)$ on phase-space, mass constraint

$$\int_{\mathbb{R}^4} m(x,p) = (2\pi)^2$$

$$\rho(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} m(x,p)dp$$

- Self-generated magnetic vector potential

$$A_\rho(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \rho(y)dy$$

- Vlasov-like energy

$$\boxed{\mathcal{E}^{\mathrm{Vla}}[m] := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \left(|p + \beta A_\rho(x)|^2 + V(x) \right) m(x,p)dxdp}$$

- Semi-classical Pauli principle

$$0 \leq m(x,p) \leq 1$$

- Minimizers of the form

$$m_\rho(x,p) = \mathbb{1}_{\left\{ |p+\beta A_\rho(x)|^2 \leq 4\pi\rho(x) \right\}} \text{ with } \mathcal{E}^{\mathrm{Vla}}[m_\rho] = \mathcal{E}^{\mathrm{TF}}[\rho] = \int_{\mathbb{R}^2} \left(2\pi\rho^2 + V\rho \right)$$

Justifying the mean-field approximation

- Thomas Fermi ground state ρ^{TF} and ground-state energy e^{TF}

$$e^{\mathrm{TF}} = \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}] = \inf \left\{ \mathcal{E}^{\mathrm{TF}}[\rho], 0 \le \rho, \int_{\mathbb{R}^2} \rho = 1 \right\}$$

- Associated Vlasov minimizer

$$m^{\mathrm{TF}}(x,p) = \mathbb{1}_{\left\{ \left| p + \beta A_{\rho^{\mathrm{TF}}}(x) \right|^2 \le 4\pi \rho^{\mathrm{TF}}(x) \right\}}$$

- Associated density in momentum space

$$t^{\mathrm{TF}}(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} m^{\mathrm{TF}}(x,p) dx$$

Theorem (Girardot, NR 2021)

In the previous limit, with $R = N^{-\eta}, \quad 0 < \eta < \frac{1}{4}$,

$$\boxed{\frac{E(N,R,\alpha)}{N} \underset{N\to\infty}{\to} e^{\mathrm{TF}}}$$

Moreover, for a ground state Ψ_N, *(*$\mathcal{F}_\hbar$ *Fourier transform)*

$$\rho^{(k)}_{\Psi_N} := \int_{\mathbb{R}^{2(N-k)}} |\Psi_N|^2 \underset{N\to\infty}{\to} \left(\rho^{\mathrm{TF}}\right)^{\otimes k}, \quad t^{(k)}_{\Psi_N} := \int_{\mathbb{R}^{2(N-k)}} |\mathcal{F}_\hbar(\Psi_N)|^2 \underset{N\to\infty}{\to} \left(t^{\mathrm{TF}}\right)^{\otimes k}$$

Numerics for quasi-fermions

- Hartree approximation, intermediate between many-body and Vlasov
- Slater determinant $\Psi_N = \det(u_j(x_k))$ as ansatz, neglect exchange terms
- Density matrix/density/self-consistent vector potential

$$\gamma(x;y) = \sum_{j=1}^{N} u_j(x)\overline{u_j(y)} \quad \rho(x) = \sum_{j=1}^{N} u_j(x)\overline{u_j(y)}$$

- Mean-field energy

$$\mathcal{E}_\alpha^{\mathrm{H}}[\gamma] := \mathrm{Tr}\left(\left(-\mathrm{i}\nabla + A[\rho]\right)^2 \gamma\right) + N\int_{\mathbb{R}^2} V\rho$$

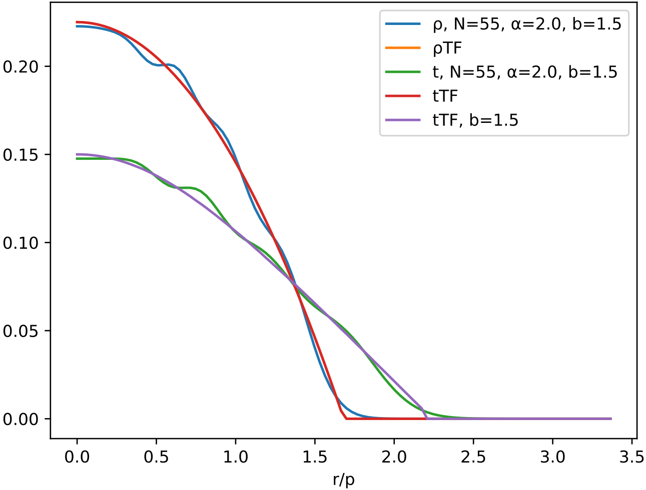
- Ground-state problem

$$E^{\mathrm{H}}(N,\alpha) := \inf\left\{\mathcal{E}_\alpha^{\mathrm{H}}[\gamma], 0 \leq \gamma \leq 1, \mathrm{Tr}\gamma = N, \gamma \text{ rank } N\right\}.$$

- Anyonic signatures only in the momentum distribution

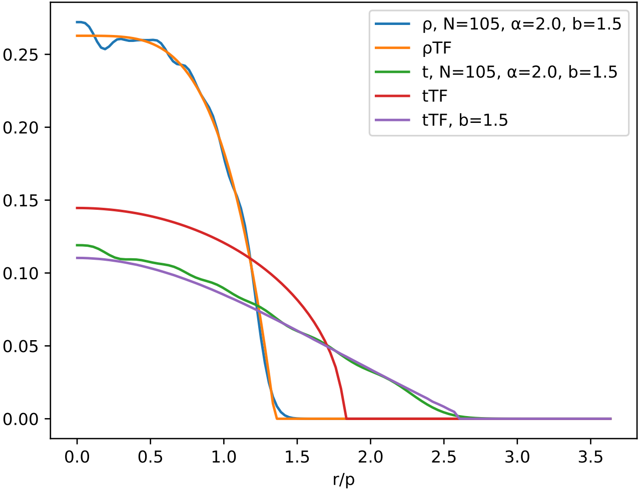
$$\widehat{\gamma}(p;q) = \frac{1}{(2\pi\hbar)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \gamma(x;y) e^{-\mathrm{i}\frac{p\cdot x - q\cdot y}{\hbar}} dxdy.$$

Numerics for quasi-fermions



$$V(x) = |x|^2, \beta = 1.5$$

Numerics for quasi-fermions



$$V(x) = |x|^4, \beta = 1.5$$

Conclusions

- Anyons are an exotic possibility of 2D many-body quantum mechanics
- Violation of the ubiquituous boson/fermion dichotomy !!
- They have staid hidden by quantization subtleties for years
- They can be realized as quasi-particles of the FQHE
- Derivation and analysis of effective models of anyons is crucial to interpret future (?) experiments

Prospects

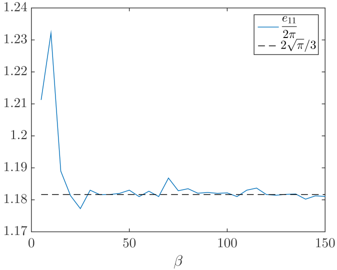
- Mean-field limit: dynamics, dilute systems ...
- Magnetic NLS model: prove the occurence of vortices.
- Other mean-field limits: at fixed α, obtain a *magnetic* Thomas-Fermi model
- Investigate the fixed α regime numerically, with Hartree/Hartree-Fock
- Rigorously prove the emergence of the anyon Hamiltonian (work in progress on particular case, with G. Lambert and D. Lundholm)

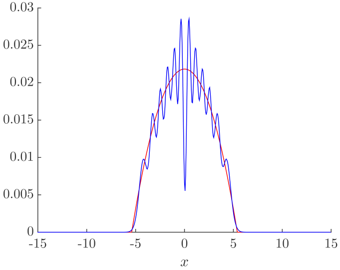
Densities: theory/numerics

- Recall the Thomas-Fermi model for large β limit, exactly solvable

$$\mathcal{E}_\beta^{\mathrm{TF}}[\rho] = \int_{\mathbb{R}^2} \left(V\rho + \beta e(1,1)\rho^2 \right)$$

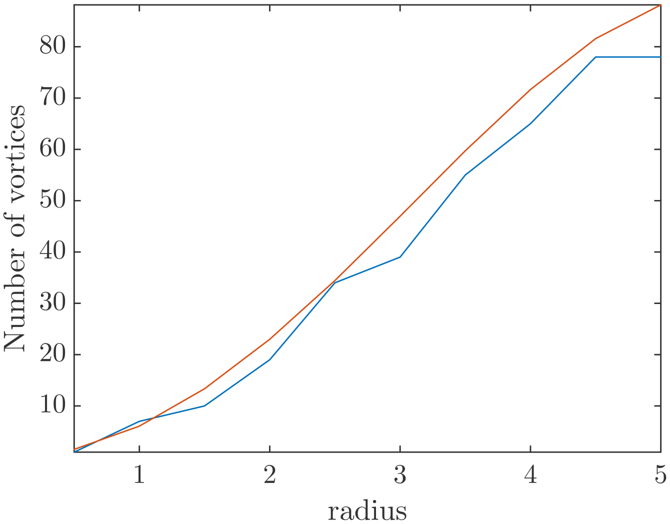
- $e(1,1)$ extracted from numerically computed energies
- Used to compare theoretical and numerical energies





$V(x) = |x|^2, \beta = 90$

Vorticities, theory/numerics



$V(x) = |x|^2, \beta = 90$