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A bulk gap in the presence of edge states for a Haldane pseudopotential

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Background: Haldane Pseudopotentials

Haldane pseudopotentials were originally introduced as Hamiltonian models for the fractional quantum Hall effect for $\nu = 1/(p+2)$ with $p \geq 0$ odd.

- Laughlin '83: Ansatz for many-body ground state wave function Ψ_p.
- Haldane '83: Pseudopotential $W^p \geq 0$ obtained via projection onto lowest Landau level of repulsive, short-range, radially symmetric pair potential:

$$W^p = \sum_{i<j} P_{LLL} v_p(z_i - z_j) P_{LLL}, \quad v_p \propto \Delta^p \delta, \quad z = x + iy$$

  Tailored so $\Psi_p \in \ker W^p$.
- Haldane-Rezayi '85, Trugman-Kivelson '85, Lee-Papic-Thomale '17, ...: More generalized study of various pseudopotentials on different 2D geometries.
- Regnault-Jolicoeur '04, Cooper '08,...: Also model rapidly rotating Bose gases (p even).
- Lewin-Seiringer '09, Seiringer-Yngvason '20: Obtained as scaling limit.
- Johri-Papic-Schmitteckert-Bhatt-Haldane '12: Properties of pseudopotentials robust under change of geometry.

Properties and Conjectures:

$$\Lambda = \text{ physical space}, \quad N = \text{ number of particles}, \quad \nu = \frac{N}{|\Lambda|} \text{ filling factor}$$

1. Ground States: Zero energy states $\psi \in \mathcal{G}_\Lambda = \ker W_\Lambda^p$ satisfy $\nu \le \nu(p) := \dfrac{1}{p+2}$.
   - E.g. Ψ_p has maximal filling $\nu(p)$.
2. Spectral Rigidity: For states with higher fillings $\nu > \nu(p)$:

$$E_0(\mathcal{H}_\Lambda^N) = \inf_{0 \neq \psi \in \mathcal{H}_\Lambda^N} \frac{\langle \psi | W_\Lambda^p \psi \rangle}{\|\psi\|^2} \propto |\Lambda| \times \text{ increasing function of } \nu$$

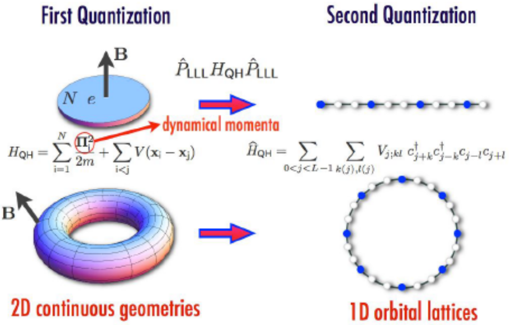
   - Determines Yrast line for Bose gases: Viefers-Hansson-Reimann '00, Regnault-Jolicoeur '04, Lewin-Seiringer '09,...
3. Spectral Gap Conjecture: Haldane '83, Haldane-Rezayi '85, ...

$$\gamma := \inf_\Lambda \mathrm{gap}(W_\Lambda^p) > 0 \quad \text{where} \quad \mathrm{gap}(W_\Lambda^p) = \inf_{0 \neq \psi \perp \mathcal{G}_\Lambda} \frac{\langle \psi | W_\Lambda^p \psi \rangle}{\|\psi\|^2}$$

   - The gap is responsible for the incompressibility of the FQH fluid: $E_0(\mathcal{H}_\Lambda^N) = 0$ for $\nu \le \nu(p)$ and $E_0(\mathcal{H}_\Lambda^N) > \gamma$ for $\nu > \nu(p)$.

4 Anyonic Excitations with Fractional Charge and their topological stability: Hastings-Michalakis '15, Haah '16, Cha-Naaijkens-Nachtergaele '20,...

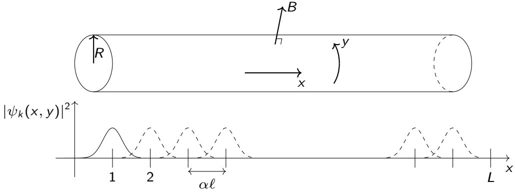
The 1/3-Haldane Pseudopotential



In second quantization, the pseudopotentials become one-dimensional (orbital) lattice models. Lee-Leinaas '04, Bergholtz-Karlhede, '05, Nakamura-Wang-Bergholtz '12,...

Today: Consider the low-lying spectral properties of a truncated version of the lattice model for the 1/3-pseudopotential ($p = 1$) in the cylinder geometry.

The 1/3-Haldane Pseudopotential



magnetic length: ℓ

ratio: $\alpha = \ell/R$

Figure: The Landau orbitals. A magnetic flux $2\pi\beta$ along the cylinder axis shifts the orbitals by $\beta\alpha\ell$ (not shown).

Landau orbitals: Single particle Hilbert space is $\mathcal{H}_{LLL} = \mathrm{span}\{\psi_k | k \in \mathbb{Z}\}$ where

$$\psi_k(x,y) \propto \exp\left(ik\frac{\alpha y}{\ell}\right)\exp\left(-\frac{1}{2}\left[\frac{x}{\ell} - k\alpha\right]^2\right).$$

W^1: Projection of $v_1 \propto \Delta\delta$ onto fermionic Fock space $\mathcal{F} = \bigoplus_{N\geq 0} S_-(\mathcal{H}_{LLL}^{\otimes N})$:

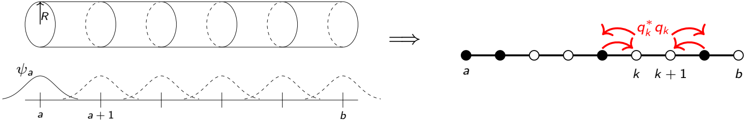
$$W^1 = \sum_{s\in\mathbb{Z}/2} B_s^* B_s, \quad B_s = \sum_{k\in s+\mathbb{Z}} ke^{-(\alpha k)^2} c_{s-k}c_{s+k}$$

where c_k is the annihilation operator for ψ_k.

Truncated model: We consider the model restricted to $|k| \leq 3/2$:

$$s \in \mathbb{Z}: \ \tilde{B}_s = 2e^{-\alpha^2} c_{s-1}c_{s+1}, \qquad s \in \mathbb{Z} + \frac{1}{2}: \ \tilde{B}_s = e^{-\frac{\alpha^2}{4}}\left(c_{s-\frac{1}{2}}c_{s+\frac{1}{2}} + 3e^{-2\alpha^2} c_{s-\frac{3}{2}}c_{s+\frac{3}{2}}\right)$$

The Finite-Volume Truncated 1/3-Model



Finite volume model: Fix $\lambda \in \mathbb{C}$ and $\kappa > 0$. For any $\Lambda = [a, b] \subseteq \mathbb{Z}$

$$H_\Lambda = \sum_{k=a}^{b-2} n_k n_{k+2} + \kappa \sum_{k=a+1}^{b-2} q_k^* q_k$$

$$n_k = c_k^* c_k, \quad q_k = c_k c_{k+1} - \lambda c_{k-1} c_{k+2}$$

$$\mathcal{H}_\Lambda = \mathrm{span}\left\{|\mu_a, \ldots, \mu_b\rangle : \mu_k \in \{0, 1\}\right\}, \qquad \mu_k = \text{ occupation of } \psi_k$$

Symmetries:

Particle number: $N_\Lambda = \sum_{k=a}^{b} n_k$, Center of mass: $M_\Lambda = \sum_{k=a}^{b} k n_k$

Physical regime: $\kappa = \dfrac{e^{3\alpha^2/2}}{4}$ and $\lambda = -3e^{-2\alpha^2}$ where $\alpha = \frac{\ell}{R}$.

Tao-Thouless limit: $\lambda \to 0$ as $R \to 0$.

Uniform Spectral Gap (OBC)

$\min \mathrm{spec}(H_\Lambda) = 0$ for any interval $\Lambda = [a, b]$. Thus, the ground state gap is

$$\mathrm{gap}(H_\Lambda) := \min\{E \in \mathrm{spec}(H_\Lambda) : E > 0\}.$$

The system has a uniform spectral gap if there is some $L_0 > 0$ so that

$$\gamma := \inf_{L \geq L_0} \mathrm{gap}(H_{[-L,L]}).$$

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Edge Modes for $0 < |\lambda| << 1$: Lowest eigenvalue of H_Λ is $\frac{\kappa}{\kappa+1}|\lambda|^2 + \mathcal{O}(|\lambda|^4)$ in invariant subspace $\operatorname{span}\{|110010\ldots0\rangle, |101100\ldots0\rangle\}$

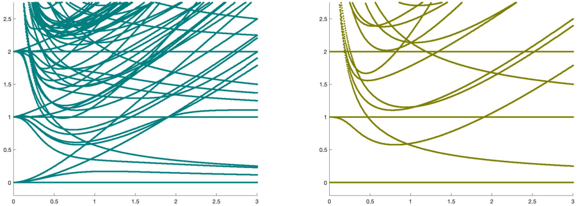


Figure: Plot of the spectrum for $H_{[1,9]}(\lambda)$ and $H^{\mathrm{per}}_{[1,9]}(\lambda)$, resp., for 1/3-truncated model in physical regime.

Bulk Gap (PBC)

Consider the same gap question for the model with periodic boundary conditions:

$$H_\Lambda^{\mathrm{per}} = \sum_{k=a}^{b} n_k n_{k+2} + \kappa q_k^* q_k$$

where $b+1 \equiv a$. Still have $\min \mathrm{spec}(H_\Lambda^{\mathrm{per}}) = 0$. The uniform periodic gap

$$\gamma^{\mathrm{per}}(\kappa, \lambda) := \inf_{L \geq L_0} \mathrm{gap}(H_{[-L,L]}^{\mathrm{per}}) > 0$$

should be $\mathcal{O}(1)$ as $\lambda \to 0$. This is one way to characterize a bulk gap.

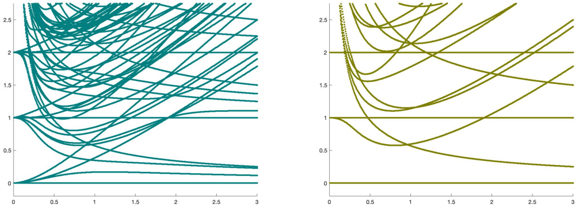


Figure: Plot of the spectrum for $H_{[1,9]}(\lambda)$ and $H_{[1,9]}^{\mathrm{per}}(\lambda)$, resp., for 1/3-truncated model in physical regime.

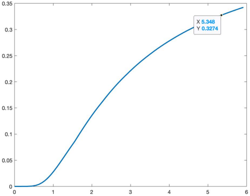
Problem: Methods for estimating $\mathrm{gap}(H_\Lambda^{\mathrm{per}})$ rely on $\mathrm{gap}(H_{\Lambda'})$ for $\Lambda' \subseteq \Lambda$.

Main Result: Bulk Spectral Gap

Edge Modes for OBC: Lowest eigenvalue of H_Λ is $\frac{\kappa}{\kappa+1}|\lambda|^2 + \mathcal{O}(|\lambda|^4)$ in invariant subspace $\mathrm{span}\{|110010\ldots0\rangle, |101100\ldots0\rangle\}$

$$\gamma^{\mathrm{obc}} = \frac{1}{4}\min\left\{3\gamma^{\mathrm{per}}, \frac{\kappa|\lambda|^2}{2\kappa+1}\right\}$$

$$\gamma^{\mathrm{per}} = \frac{1}{3}\min\left\{1, \frac{\kappa}{\kappa+1}, \frac{\kappa}{2+2\kappa|\lambda|^2}\right\}$$



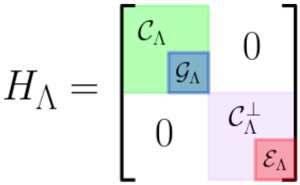
Theorem: [Warzel, Y. '21] *For all $\lambda \neq 0$ with $f(|\lambda|^2) < 1/3$ (i.e. $|\lambda| < 5.3\ldots$)*

- **OBC gap:** $\inf_{|\Lambda|\geq 10} \mathrm{gap}(H_\Lambda) \geq \min\left\{\gamma^{\mathrm{obc}}, \frac{\kappa}{3}\left(1-\sqrt{3f(|\lambda|^2)}\right)^2\right\}$
- **Bulk gap:** $\liminf_{|\Lambda|\to\infty} \mathrm{gap}(H_\Lambda^{\mathrm{per}}) \geq \min\left\{\gamma^{\mathrm{per}}, \frac{\kappa}{6(1+2|\lambda|^2)}\left(1-\sqrt{3f(|\lambda|^2)}\right)^2\right\}$

Remarks:

- Bulk gap stays open despite edge states for OBC. Strengthens result from [Nachtergaele, Warzel, Y. '21].
- Analogous strategy/result holds for $p = 0$ model [Warzel, Y. '21].

Block-Diagonalization Strategy



For both $\# \in \{\mathrm{obc}, \mathrm{per}\}$ decompose $\mathcal{H}_\Lambda = \mathcal{C}_\Lambda^\# \oplus (\mathcal{C}_\Lambda^\#)^\perp$ so that:

- Invariant under $H_\Lambda^\#$: $H_\Lambda^\# \mathcal{C}_\Lambda^\# \subseteq \mathcal{C}_\Lambda^\#$
- Contains ground state space: $\mathcal{G}_\Lambda^\# := \ker(H_\Lambda^\#) \subseteq \mathcal{C}_\Lambda^\#$
- Separates edge states: Edge states of H_Λ^{obc} are contained in $(\mathcal{C}_\Lambda^{\mathrm{obc}})^\perp \subseteq (\mathcal{C}_\Lambda^{\mathrm{per}})^\perp$.

As a consequence:

$$\mathrm{gap}(H_\Lambda^\#) = \min\left\{E_1(\mathcal{C}_\Lambda^\#),\ E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big)\right\} \quad \text{where}$$

$$E_1(\mathcal{C}_\Lambda^\#) := \inf_{\psi \in \mathcal{C}_\Lambda^\# \cap (\mathcal{G}_\Lambda^\#)^\perp} \frac{\langle \psi | H_\Lambda^\# \psi \rangle}{\|\psi\|^2}, \qquad E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big) := \inf_{\varphi \in (\mathcal{C}_\Lambda^\#)^\perp \cap \mathrm{dom}(H_\Lambda^\#)} \frac{\langle \varphi | H_\Lambda^\# \varphi \rangle}{\|\varphi\|^2}.$$

Constructing Invariant Subspaces with Ground States

Since the interaction terms are all nonnegative:

$$\ker(H_\Lambda) = \bigcap_{k=a}^{b-2} \ker(n_k n_{k+2}) \cap \bigcap_{k=a+1}^{b-2} \ker(q_k).$$

Observations:

- $|\mu\rangle$ is a ground state of the electrostatic terms iff $\mu_k \mu_{k+2} = 0$ for all k.
- $q_k = c_k c_{k+1} - \lambda c_{k-1} c_{k+2}$ acts nontrivially on the sites $[k-1, k+2]$:

$$q_k \left(|1001\rangle + \lambda |0110\rangle \right) = 0$$

A simple calculation shows $q_k^* q_k \{|1001\rangle, |0110\rangle\} \subseteq \mathrm{span}\{|1001\rangle, |0110\rangle\}$.

Starting from $|100100\ldots\rangle$, can construct a set of occupation states that span an invariant subspace of H_Λ by replacing ‘1001’ with ’0110’:

1	0	0	1	0	0	1	0	0	1	0	0
0	1	1	0	0	0	1	0	0	1	0	0
1	0	0	0	1	1	0	0	0	1	0	0
1	0	0	1	0	0	0	1	1	0	0	0
0	1	1	0	0	0	0	1	1	0	0	0

$$\psi_\Lambda(R) = \sum_{T \leftrightarrow R} \lambda^{d(T)} |T\rangle$$

[Jansen ’12], [Nakamura, Wang, Bergholtz ’12]

BVMD Tiling Spaces (OBC)

More generally, an invariant subspace of H_Λ is generated by any root tiling R of Λ consisting of **void**, **monomer**, and **boundary tiles**:

[0]   [1 0 0]   Left: [1 1 0 0 0]   Right: [1]   [1 0]   [0 1 1]

Example:

[1 1 0 0 0][1 0 0][0][1 0 0][1 0 0][0][1 0 0][1]

[1 1 0 0 0][1 0 0][0][0 1 1 0 0 0][0][1 0 0][1]
[1 1 0 0 0][1 0 0][0][1 0 0][1 0 0][0][0 1 1 0]
[1 1 0 0 0][1 0 0][0][0 1 1 0 0 0][0][0 1 1 0]

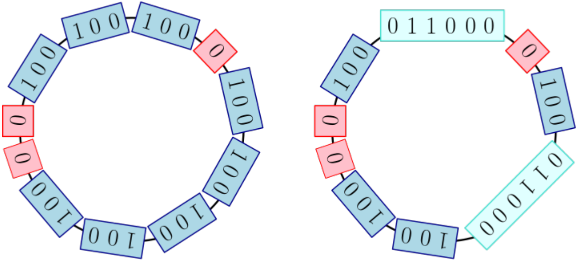
The Boundary-Void-Monomer-Dimer (BVMD) space generated by a root tiling R is

$$\mathcal{C}_\Lambda(R) = \mathrm{span}\{|T\rangle : T \leftrightarrow R\}.$$

Lemma: [Nachtergaele, Warzel, Y. '21] $\mathcal{G}_\Lambda^{\mathrm{obc}} \subseteq \mathcal{C}_\Lambda^{\mathrm{obc}} := \bigoplus_R \mathcal{C}_\Lambda(R)$. BVMD spaces generated by different roots are orthogonal, and each contains a unique ground state:

$$\psi_\Lambda(R) = \sum_{T \leftrightarrow R} \lambda^{d(T)} |T\rangle.$$

VMD Tiling Spaces (PBC)



Analogous construction in the case of periodic boundary conditions:

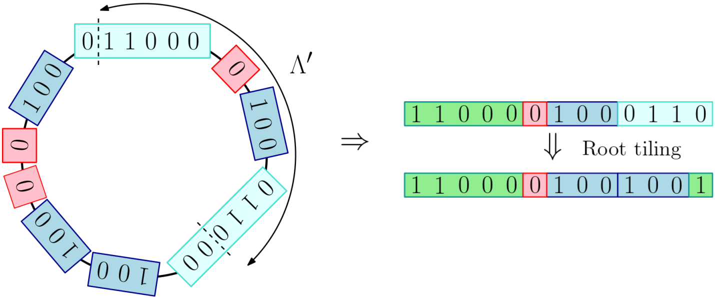
$$\mathcal{G}_\Lambda^{\mathrm{per}} \subseteq \mathcal{C}_\Lambda^{\mathrm{per}} := \bigoplus_R \mathcal{C}_\Lambda^{\mathrm{per}}(R), \quad \psi_\Lambda^{\mathrm{per}} = \sum_{T \leftrightarrow R} \lambda^{d(T)} |T\rangle,$$

where root tilings of the ring only use monomers and voids.

Properties: 1. $\dim \mathcal{G}_\Lambda^{\mathrm{per}} \propto \left(\frac{1+\sqrt{5}}{2}\right)^{|\Lambda|}$ 2. Maximum filling: $N_\Lambda(R)/|\Lambda| \leq 1/3$

Isospectral relationship

If $\Lambda' \subset \Lambda$, then $T_\Lambda \restriction_{\Lambda'} = T_{\Lambda'}$:



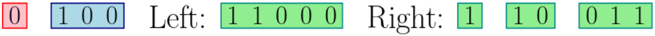
As a consequence, if $|\Lambda| \geq |\Lambda'| + 4$, then for either $\# \in \{\mathrm{obc}, \mathrm{per}\}$,

$$\mathrm{spec}(H_{\Lambda'}^{\mathrm{obc}} \restriction_{\mathcal{C}_\Lambda^\#}) = \mathrm{spec}(H_{\Lambda'}^{\mathrm{obc}} \restriction_{\mathcal{C}_{\Lambda'}^{\mathrm{obc}}}).$$

This relationship is key for computing the uniform lower bound on the spectral gap in the tiling space.

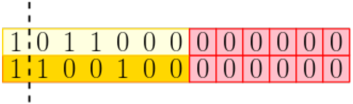
Edge Tiling Spaces (OBC)

For open boundary conditions, every state with energy $\mathcal{O}(|\lambda|^2)$ belongs to an invariant subspace generated from a root tiling consisting of the BVMD-tiles



and at least one edge boundary tile:

The edge tiling spaces require several other new tiles and replacement rules. Regardless, these tilings only differ from BVMD tilings at the first and/or last site of Λ.



Gap Estimating Strategy

Recap: For both $\# \in \{\mathrm{obc}, \mathrm{per}\}$, we have written $\mathcal{H}_\Lambda = \mathcal{C}_\Lambda^\# \oplus (\mathcal{C}_\Lambda^\#)^\perp$ where

$$\mathcal{G}_\Lambda^\# \subseteq \mathcal{C}_\Lambda^\# = \bigoplus_R \mathcal{C}_\Lambda^\#(R), \qquad (\mathcal{C}_\Lambda^\#)^\perp = \mathrm{span}\{|\mu\rangle = |\mu_a \ldots \mu_b\rangle : |\mu\rangle \notin \mathcal{C}_\Lambda^\#\}$$

are invariant under all interaction terms, and $\mathcal{E}_\Lambda \subseteq (\mathcal{C}_\Lambda^{\mathrm{obc}})^\perp \subseteq (\mathcal{C}_\Lambda^{\mathrm{per}})^\perp$. Thus,

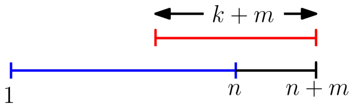
$$\mathrm{gap}(H_\Lambda^\#) = \min\left\{E_1(\mathcal{C}_\Lambda^\#),\, E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big)\right\}$$

Methods: Lower bounds uniform in $|\Lambda|$ are obtained with the following approaches:

- For $E_1(\mathcal{C}_\Lambda^\#)$, apply gap techniques previously developed for quantum spin models:
  - $E_1(\mathcal{C}_\Lambda^{\mathrm{obc}})$: martingale method [Nachtergaele, '96], [Nachtergaele, Sims, Y. '18]
  - $E_1(\mathcal{C}_\Lambda^{\mathrm{per}})$: finite size criterion [Knabe, '89] + OBC result for $E_1(\mathcal{C}_\Lambda^{\mathrm{obc}})$
- For $E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big)$, use electrostatic estimates to lower bound minimum energy.

  Important: PBC estimate does not require use of OBC estimate!

Gap Methods



Set up: Finite-range, frustration-free, translation-invariant[1] quantum lattice model on $\mathcal{H}_\Lambda$ with local ground state spaces $\mathcal{G}_{\Lambda'} \equiv \ker(H_{\Lambda'} \otimes \mathbb{1}_{\Lambda \setminus \Lambda'}) \subset \mathcal{H}_\Lambda$.

- Martingale Method: Fix $k, m \in \mathbb{N}$. If there is an $\ell > 0$ sufficiently large such that

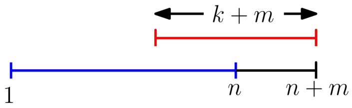
$$\epsilon := \sup_{n \geq n_0} \| G_{[n-k+1,n+m]}(\mathbb{1} - G_{[1,n+m]}) G_{[1,n]} \| < 1/\sqrt{\ell},$$

then

$$\inf_{n \geq n_0+m} \mathrm{gap}(H_{[1,n]}) \geq \frac{\mathrm{gap}(H_{[1,k+m]})}{\ell}(1-\epsilon\sqrt{\ell})^2.$$

[1] Not strictly necessary, but simplifies expressions.

Gap Methods



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then

$$\inf_{n \geq n_0 + m} \mathrm{gap}(H_{[1,n]}) \geq \frac{\mathrm{gap}(H_{[1,k+m]})}{\ell} (1 - \epsilon\sqrt{\ell})^2.$$

- Finite Size Criterion: If $\mathrm{gap}(H_{[1,n]}) > \frac{||H_{[1,k+m]}||}{n}$, then for any $|\Lambda| \geq 2n$

$$\mathrm{gap}(H_\Lambda^{\mathrm{per}}) \geq \frac{(n-1)\mathrm{gap}(H_{[1,k+m]})}{n\ell ||H_{[1,k+m]}||} \left[\mathrm{gap}(H_{[1,n]}) - \frac{||H_{[1,k+m]}||}{n} \right]$$

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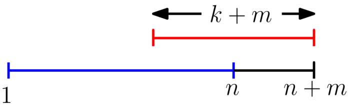
Gap Methods



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Application to FQHE: Apply to $\mathcal{H}_\Lambda = \mathcal{C}_\Lambda^{\#}$ and use isospectral relations to calculate ground state operator norm and gap bounds with $\ell, m = 3$ and $k = 6$.

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[1] Not strictly necessary, but simplifies expressions.

Bounding $E_0\big((\mathcal{C}_\Lambda^\#)^\perp\big)$

Partition non-BVMD tiling configurations $\mathcal{S}_\Lambda^\# = \{\mu : |\mu\rangle \notin \mathcal{C}_\Lambda^\#\}$ as

$$\mathcal{S}_\Lambda^\# = \mathcal{S}_E^\# \,\dot\cup\, \mathcal{S}_D^\#, \quad \mathcal{S}_E^\# = \left\{\mu \in \mathbb{N}_0^\Lambda : e_\Lambda^\#(\mu) > 0\right\}$$

where we introduce the electrostatic energies

$$e_\Lambda^{\mathrm{obc}}(\mu) = \sum_{k=a}^{b-2} \mu_k \mu_{k+2}, \quad e_\Lambda^{\mathrm{per}}(\mu) = \sum_{k=a}^{b} \mu_k \mu_{k+2}.$$

Goal: For any $\psi = \sum_{\mu \in \mathcal{S}_\Lambda^\#} \psi(\mu)|\mu\rangle \in (\mathcal{C}_\Lambda^\#)^\perp$, bound

$$\langle \psi | H_\Lambda^\# \psi \rangle = \sum_{\mu \in \mathcal{S}_E^\#} e_\Lambda^\#(\mu)|\psi(\mu)|^2 + \sum_{\nu \in \{0,1\}^\Lambda} \sum_{k \in \Lambda^\#} |\langle \nu | q_k \psi \rangle|^2 \geq \gamma^\# \sum_{\mu \in \mathcal{S}_\Lambda^\#} |\psi(\mu)|^2$$

where $\Lambda^{\mathrm{obc}} = [a+1, b-2]$ and $\Lambda^{\mathrm{per}} = [a, b]$.

Strategy: Choose individual (ν_μ, k_μ) for each $\mu \in \mathcal{S}_D^\#$ and apply CS to get desired estimate.

Bounding $E_0\big((\mathcal{C}_\Lambda^{\mathrm{obc}})^\perp\big)$

Example: $\mu = (110010\ldots0) \in \mathcal{S}_D^{\mathrm{obc}}$ (an edge state configuration).

For OBC: Choosing $k_\mu = a+2$ and $\nu_\mu = (100\ldots0)$ so that for any $0<\delta<1$

$$|\langle \nu_\mu | q_{k_\mu}\psi\rangle|^2 = |\psi(\eta) - \overline{\lambda}\psi(\mu)|^2 \geq -\frac{1-\delta}{\delta}|\psi(\eta)|^2 + |\lambda|^2(1-\delta)|\psi(\mu)|^2$$

where $\eta = (10110\ldots0) \in S_E^{\mathrm{obc}}$.

Bounding $E_0\big((\mathcal{C}_\Lambda^{\mathrm{obc}})^\perp\big)$

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where $\eta = (10110\ldots 0) \in S_E^{\mathrm{obc}}$. Picking $\delta = 2\kappa/(1+2\kappa)$ produces the estimate

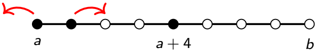
$$e_\Lambda^{\mathrm{obc}}(\eta)|\psi(\eta)|^2 + \kappa|\langle \nu_\mu | q_{k_\mu}\psi\rangle|^2 \geq \frac{1}{2}|\psi(\eta)|^2 + \frac{\kappa|\lambda|^2}{1+2\kappa}|\psi(\mu)|^2.$$

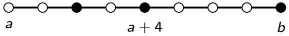
This reflects $\gamma^{\mathrm{obc}} = \mathcal{O}(|\lambda|^2)$ and goes to zero in Tao-Thouless limit!

General Strategy: Systematically chose a unique (ν_μ, k_μ) for each $\mu \in \mathcal{S}_D^{\mathrm{obc}}$ so that don't "eat" all electrostatic energy of any given $\eta \in \mathcal{S}_E^{\mathrm{obc}}$.

Bounding $E_0\big((\mathcal{C}_\Lambda^{\mathrm{per}})^\perp\big)$

Example: $\mu = (11001000\ldots0) \in \mathcal{S}_D^{\mathrm{per}}$ (same configuration).





For PBC: For same μ, choosing $k_\mu = a$ with $\nu_\mu = (000010\ldots0)$ produces:

$$|\langle \nu_\mu | q_{k_\mu} \psi \rangle|^2 = |\psi(\mu) - \lambda\psi(\eta)|^2$$

and same strategy with $\delta = \frac{2\kappa|\lambda|^2}{1+2\kappa|\lambda|^2}$ yields:

$$e_\Lambda^{\mathrm{per}}(\eta)|\psi(\eta)|^2 + \kappa|\langle \nu_\mu | q_{k_\mu}\psi\rangle|^2 \geq \frac{1}{2}|\psi(\eta)|^2 + \frac{2\kappa}{1+2\kappa|\lambda|^2}|\psi(\mu)|^2$$

where $\eta = (001010\ldots01)$. Recall in physical regime $\kappa = \mathcal{O}(1)$ when $\lambda << 1$.

Conclusion:

Summary and Auxiliary Results:

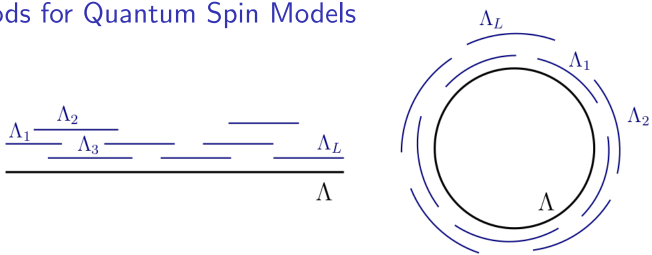
1. Proved conjectured ground state energy properties of truncated pseudopotential in a cylinder geometry facilitated by invariant subspaces described in terms of tilings.
2. A bulk gap strategy: approach valid for other models where edge states and ground states can be separated into different invariant subspaces.
3. Modified domino tilings also used to identify invariant subspaces conjectured to contain first and second excited energy states for $|\lambda| << 1$. Supported by numerical evidence.

Interesting Questions and Future Directions:

1. Better control of first and second excited states. Low complexity?
2. Longer range truncations? Stability of the gap?
3. The untruncated Haldane model.

Thank you for your attention!

Gap Methods for Quantum Spin Models



- For $\dim(\mathcal{H}_\Lambda) < \infty$ and frustration-free model: $\mathcal{G}_{\Lambda'} \equiv \ker(H_{\Lambda'} \otimes \mathbb{1}_{\Lambda \setminus \Lambda'}) \ \ \forall \Lambda' \subseteq \Lambda$.
- Requires a covering of Λ by a sequence of smaller intervals with OBC.

$$\gamma = \inf_i \operatorname{gap}(H_{\Lambda_i}), \qquad \Gamma = \sup_i \|H_{\Lambda_i}\|, \qquad \Lambda_{n,k} = \bigcup_{i=k}^{n-k+1} \Lambda_i$$

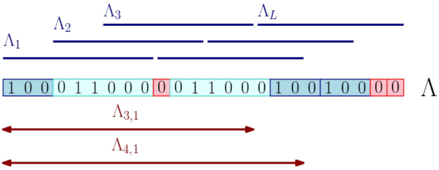
- Martingale Method: If $\epsilon := \sup_n \|G_{\Lambda_{n+1}}(\mathbb{1} - G_{\Lambda_{n+1,1}})G_{\Lambda_{n,1}}\| < 1/\sqrt{\ell}$, then

$$\operatorname{gap}(H_\Lambda) \geq \frac{\gamma}{\ell}(1 - \epsilon\sqrt{\ell})^2.$$

- Finite Size Criterion: For any n such that $|\Lambda_{n,k}| < |\Lambda|$ for all k:

$$\operatorname{gap}(H_\Lambda^{\mathrm{per}}) \geq \frac{\gamma n}{\ell\Gamma(n-1)} \left[\inf_k \operatorname{gap}(H_{\Lambda_{n,k}}) - \frac{\Gamma}{n} \right]$$

Bounding $E_1(\mathcal{C}_\Lambda^{\mathrm{obc}})$



We apply the martingale method to the Hilbert space $\mathcal{C}_\Lambda^\infty$ and Hamiltonians

$$H_{\Lambda'}^{\mathrm{bulk}} := (H_{\Lambda'} \otimes \mathbb{1}_{\Lambda \setminus \Lambda'}) \restriction_{\mathcal{C}_\Lambda^{\mathrm{obc}}} \quad \forall \Lambda' \subseteq \Lambda$$

where the overlap intervals are chosen so that $\ell = 3$ and $|\Lambda_n| = 9$ for all n.

Calculations using isospectrality and orthogonality of the BVMD states/spaces gives

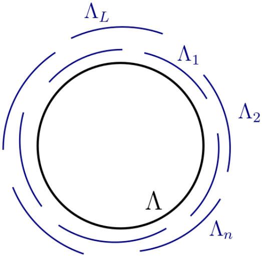
$$\gamma = \mathrm{gap}(H_{[1,9]}^{\mathrm{bulk}}) = \mathrm{gap}(H_{[1,9]} \restriction_{\mathcal{C}_{[1,9]}^{\mathrm{obc}}}) = \kappa$$

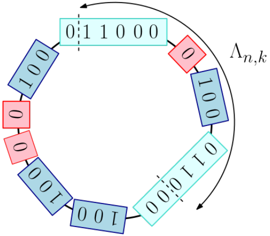
$$\epsilon = \sup_n \| G_{[3n-5,3n+3]}^{\mathrm{bulk}} (\mathbb{1} - G_{[1,3n+3]}^{\mathrm{bulk}}) G_{[1,3n]}^{\mathrm{bulk}} \| \leq \sqrt{f(|\lambda|^2)}$$

where $\mathcal{G}_{\Lambda'}^{\mathrm{bulk}} = \ker(H_{\Lambda'}^{\mathrm{bulk}}) \subseteq \mathcal{C}_\Lambda^{\mathrm{obc}}$. This produces the final estimate:

$$E_1(\mathcal{C}_\Lambda^{\mathrm{obc}}) \geq \frac{\kappa}{3} \left(1 - \sqrt{3 f(|\lambda|^2)} \right)^2 .$$

Bounding $E_1(\mathcal{C}_\Lambda^{\mathrm{per}})$





We apply Knabe's finite size criteria to the Hilbert space $\mathcal{C}_\Lambda^{\mathrm{per}}$ and Hamiltonians

$$H_\Lambda^{\mathrm{per}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}} \quad \text{and} \quad H_{\Lambda_{n,k}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}$$

where $\Lambda_{n,k} = \bigcup_{i=k}^{n+1-k} \Lambda_i$ chosen so $|\Lambda_i| = 5, 6$. Another isospectral argument shows

$$\|H_{\Lambda_i} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}\| = \kappa(1 + 2|\lambda|^2) \quad \text{and} \quad \mathrm{gap}(H_{\Lambda_{n,k}} \restriction_{\mathcal{C}_\Lambda^{\mathrm{per}}}) \geq E_1(\mathcal{C}_{\Lambda_{n,k}}^{\mathrm{obc}})$$

This yields:

$$E_1(\mathcal{C}_\Lambda^{\mathrm{per}}) \geq \frac{n}{2(n-1)(1+2|\lambda|^2)} \left(E_1(\mathcal{C}_{\Lambda_{n,k}}^{\mathrm{obc}}) - \frac{\kappa(1+2|\lambda|^2)}{n} \right)$$